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On some variations of the irregularity

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Abstract

Let G be a finite simple graph with the vertex set V, edge set E and minimum degree at least 1. For any non-zero real number α, the general ordinary irregularity $irr_{o,\alpha}$ and general total irregularity $irr_{t,\alpha}$ for the graph G are defined as $irr_{o,\alpha} = \sum_{uv\in E} |deg(u)^{\alpha} - deg(v)^{\alpha}|$ and $irr_{t,\alpha} = \sum_{\{u,v\}\subseteq V} |deg(u)^{\alpha} - deg(v)^{\alpha}|$, respectively. We denote the graph parameter $irr_{t,\alpha} - irr_{o,\alpha}$ by $\overline{irr}_{o,\alpha}$ and call it general ordinary co-irregularity. In this paper, some mathematical aspects of the parameters $irr_{o,\alpha}$, $irr_{t,\alpha}$ and $\overline{irr}_{o,\alpha}$ are explored for $\alpha = 1, 2$. Graphs with the first eight smallest $irr_{o,2}$-values are also characterized from the class of all n-vertex trees.

Keywords: Irregularity; total irregularity; general ordinary irregularity; general total irregularity; general ordinary co-irregularity.

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1. Introduction

This paper is concerned with only finite and simple graphs. The vertex set and edge set of a graph G is denoted by $V(G)$ and $E(G)$, respectively. We use the notation $deg_G(v)$ and $N_G(v)$ to denote the degree of a vertex $v \in V(G)$ and the set of all vertices adjacent to v, respectively. The symbol $\Delta(G)$ is used for the maximum degree of G. The number of vertices of degree i are denoted by $n_i(G)$. The number of those edges in a graph G whose end vertices have degrees i and j is denoted by $m_{i,j}(G)$. We will drop the letter "G" from the aforementioned notations when there is no chance of confusion.

The set of all degrees of the vertices in a graph is called degree set. A graph whose degree set consists of only one element is referred as a regular graph. By a nonregular graph, we mean a graph which is not regular. An irregularity measure IM of a graph G is a non-negative graph parameter satisfying the property: $IM(G) = 0$ if and only if every component of G is regular. The main purpose of an irregularity measure is actually to check how much a graph is nonregular with respect to the considered irregularity measure. In the paper [3], Albertson introduced the following irregularity measure and named it as "irregularity":

$$irr(G) = \sum_{uv\in E} |deg(u) - deg(v)|.$$

The irregularity measure irr is also called Albertson index [19]. The concept of irr has been modified in several directions. More precisely, Abdo, Brandt and Dimitrov [1] devise the following extended version of the irregularity measure irr in order to overcome its certain shortcomings and they named it as the "total irregularity":

$$irr_t(G) = \sum_{\{u,v\}\subseteq V} |deg(u) - deg(v)|.$$

The following extensions of the irregularity measure irr was considered by Gutman [8]:

$$irr_{Alb2}(G) = \sum_{uv\in E} (deg(u) - deg(v))^2 \quad \text{and} \quad irr_{tot2}(G) = \sum_{\{u,v\}\subseteq V} (deg(u) - deg(v))^2.$$

The measure irr_{Alb2} is known as the sigma index [2, 11, 16]. Recently, Yousaf *et al.* [19] studied the following modified version of irr and called it as the "modified Albertson index":

$$A^*(G) = \sum_{uv\in E} |deg(u)^2 - deg(v)^2|.$$

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Recent results about the irregularity measure irr and its variants can be found in the papers [5, 7, 9, 13–15, 17, 18]. In the present study, we are concerned with the following natural generalizations of the irregularity measures irr and irr_t:

$$irr_{o,\alpha} = \sum_{uv\in E} |deg(u)^\alpha - deg(v)^\alpha|$$

and

$$irr_{t,\alpha} = \sum_{\{u,v\}\subseteq V} |deg(u)^\alpha - deg(v)^\alpha|\,,$$

where α is a non-zero real number and the graph G has minimum degree at least 1. We propose to call the measures $irr_{o,\alpha}$ and $irr_{t,\alpha}$ as the "general ordinary irregularity" and "general total irregularity". It is clear that $irr_{o,1} = irr$, $irr_{o,2} = A^*$ and $irr_{t,1} = irr_t$. We denote the graph parameter $irr_{t,\alpha} - irr_{o,\alpha}$ by $\overline{irr}_{o,\alpha}$ and call it as the "general ordinary co-irregularity". It should be mentioned here that $\overline{irr}_{o,1}(G) = irr(\overline{G})$, where $\overline{G}$ is the complement of a graph G, which is a graph with the vertex set $V(\overline{G}) = V(G)$ and two vertices in $\overline{G}$ are adjacent if and only if they are not adjacent in G. The main purpose of the present paper is to derive some relations between the parameters $irr_{o,\alpha}$, $irr_{t,\alpha}$ and $\overline{irr}_{o,\alpha}$ for $\alpha = 1, 2$. We also characterize the graphs with the first eight smallest $irr_{o,2}$ values from the class of all n-vertex trees.

2. Main results

Firstly, we establish some relations between the general ordinary irregularity $irr_{o,\alpha}$, general total irregularity $irr_{t,\alpha}$ and general ordinary co-irregularity for $\alpha = 1, 2$.

Theorem 2.1. *If $\overline{G}$ is the complement of an n-vertex graph G, then*

$$irr_{t,2}(G) + irr_{t,2}(\overline{G}) = 2(n-1)irr_t(G).$$

Proof. Because of the fact $|deg_G(u) - deg_G(v)| = |deg_{\overline{G}}(u) - deg_{\overline{G}}(v)|$, we have

$$\begin{aligned}
irr_{t,2}(G) + irr_{t,2}(\overline{G}) &= \sum_{\{u,v\}\subseteq V(G)} \big(deg_G(u) + deg_G(v)\big)|deg_G(u) - deg_G(v)| \\
&+ \sum_{\{u,v\}\subseteq V(G)} \big(deg_{\overline{G}}(u) + deg_{\overline{G}}(v)\big)|deg_G(u) - deg_G(v)| \\
&= \sum_{\{u,v\}\subseteq V(G)} |deg_G(u) - deg_G(v)|\big[deg_G(u) + deg_{\overline{G}}(u) \\
&+ deg_G(v) + deg_{\overline{G}}(v)\big] \\
&= 2(n-1) \sum_{\{u,v\}\subseteq V(G)} |deg_G(u) - deg_G(v)| = 2(n-1)irr_t(G).
\end{aligned}$$

□

Bearing in mind the inequality

$$|irr_{t,2}(G) - irr_{t,2}(\overline{G})| \le irr_{t,2}(G) + irr_{t,2}(\overline{G}),$$

we have the next corollary as a direct consequence of Theorem 2.1.

Corollary 2.1. *If $\overline{G}$ is the complement of an n-vertex graph G, then*

$$\left|irr_{t,2}(G) - irr_{t,2}(\overline{G})\right| \le 2(n-1)irr_t(G)\,.$$

Theorem 2.2. *If $\overline{G}$ is the complement of an n-vertex graph G, then*

$$irr_{o,2}(G) + \overline{irr}_{o,2}(\overline{G}) = 2(n-1)irr(G).$$

Proof. We note that

$$\begin{aligned}
irr_{o,2}(G) + \overline{irr}_{o,2}(\overline{G}) &= \sum_{uv\in E(G)} |deg_G(u)^2 - deg_G(v)^2| + \sum_{uv\notin E(\overline{G})} |deg_{\overline{G}}(u)^2 - deg_{\overline{G}}(v)^2| \\
&= \sum_{uv\in E(G)} |deg_G(u)^2 - deg_G(v)^2| + \sum_{uv\in E(G)} |deg_{\overline{G}}(u)^2 - deg_{\overline{G}}(v)^2|
\end{aligned}$$

$$= \sum_{uv\in E(G)} |deg_G(u) - deg_G(v)|\Big[deg_G(u) + deg_{\overline{G}}(u)$$
$$+ deg_G(v) + deg_{\overline{G}}(v)\Big]$$
$$= 2(n-1)irr(G),$$

as desired. □

From Theorem 2.2 and the inequality

$$|irr_{o,2}(G) - \overline{irr}_{o,2}(\overline{G})| \leq irr_{o,2}(G) + \overline{irr}_{o,2}(\overline{G}),$$

the next corollary follows.

Corollary 2.2. *If $\overline{G}$ is the complement of an n-vertex graph G, then*

$$|irr_{o,2}(G) - \overline{irr}_{o,2}(\overline{G})| \leq 2(n-1)irr(G).$$

By an (r,s)-bidegreed graph, we mean a graph with the degree set $\{r,s\}$. The path graph with n vertices is denoted by P_n.

Proposition 2.1. *Let G be a connected n-vertex nonregular graph with $n \geq 3$. Then,*

$$3irr(G) \leq irr_{o,2}(G) \leq (2n-3)irr(G), \tag{1}$$
$$3\overline{irr}(G) \leq \overline{irr}_{o,2}(G) \leq (2n-3)\overline{irr}(G), \tag{2}$$
$$3irr_t(G) \leq irr_{t,2}(G) \leq (2n-3)irr_t(G). \tag{3}$$

If $G \cong P_n$ then the left equality sign in either of Inequalities (1), (2), (3) holds and if G is an $(n-2, n-1)$-bidegreed graph then the right equality sign in either of Inequalities (1), (2), (3) holds.

Proof. We note that if $u, v \in V(G)$ such that $deg(u) \neq deg(v)$, then it holds that

$$3 \leq deg(u) + deg(v) \leq 2n-3$$

with left equality if and only if one of $deg(u)$ and $deg(v)$ is 1 and the other is 2, while the right equality holds if and only if one of $deg(u)$, $deg(v)$ is $n-1$ and the other is $n-2$. Hence, the result follows from the following inequality

$$3|deg(u) - deg(v)| \leq |deg(u)^2 - deg(v)^2| \leq (2n-3)|deg(u) - deg(v)|\,.$$

□

If the graph G is triangle-free in Proposition 2.1, then the upper bound given in (1) can be improved.

Corollary 2.3. *Let G be a connected nonregular triangle-free graph with n vertices such that $n \geq 3$. Then,*

$$irr_{o,2}(G) \leq n \cdot irr(G), \tag{4}$$

If G is an $(n-\Delta, \Delta)$-bidegreed graph then the equality sign in (4) holds.

Proof. The result follows from the fact that $deg(u) + deg(v) \leq n$ for every $uv \in E(G)$ because G is a triangle-free graph. □

Now, we derive two lower bounds on the graph parameter $irr_{t,2}$ for nonregular graphs. First such lower bound is actually in terms of n_1 and M_1, where n_i is the number of those vertices of a graph G which have degree i and M_1 is the first Zagreb index [4,6], which was firstly appeared in a formula reported in [12]. The first Zagreb index for a graph G is defined as

$$M_1 = M_1(G) = \sum_{v\in V} deg(v)^2.$$

A vertex of degree 1 is called a *pendant vertex*.

Theorem 2.3. *If G is an n-vertex nonregular graph with the first Zagreb index M_1 and with the number of pendant vertices n_1, then*

$$irr_{t,2}(G) \geq n_1(M_1 - n)$$

with equality if and only if G is a $(1, \Delta)$-bidegreed graph.

Proof. If P is the set of all pendant vertices of G, then

$$\begin{aligned} irr_{t,2}(G) &= \sum_{\{u,v\} \subseteq V} |deg(u)^2 - deg(v)^2| \\ &= \sum_{u \in V \setminus P} n_1 |deg(u)^2 - 1| + \sum_{\{u,v\} \subseteq V \setminus P} |deg(u)^2 - deg(v)^2| \\ &\geq n_1 \sum_{u \in V \setminus P} (deg(u)^2 - 1) \\ &= n_1[M_1 - n_1 - (n - n_1)] = n_1(M_1 - n). \end{aligned}$$

Clearly, the equation $irr_{t,2}(G) = n_1(M_1 - n)$ holds if and only if G is an $(1, \Delta)$-bidegreed graph. □

By an (a, b, c)-tridegreed graph, we mean a graph with the degree set $\{a, b, c\}$. Now, we derive a lower bound on $irr_{t,2}$ for nonregular graphs in terms of number of vertices and number of vertices of degrees 1 and 2.

Theorem 2.4. *If G is an n-vertex nonregular graph with n_1 and n_2 as the number of vertices of degrees 1 and 2, respectively, then*

$$irr_{t,2}(G) \geq 8nn_1 + 5nn_2 - 8n_1^2 - 5n_2^2 - 10n_1n_2$$

with equality if and only if G is either $(1, 2, 3)$-tridegreed graph or $(1, 2)$-bidegreed graph.

Proof. By definition of the graph parameter $irr_{t,2}$, we have

$$\begin{aligned} irr_{t,2}(G) &= \sum_{1 \leq i < j \leq n-1} n_i n_j (j^2 - i^2) \\ &= 3n_1n_2 + \sum_{j=3}^{n-1} n_1 n_j (j^2 - 1) + \sum_{j=3}^{n-1} n_2 n_j (j^2 - 4) + \sum_{3 \leq i < j \leq n-1} n_i n_j (j^2 - i^2) \\ &\geq 3n_1n_2 + \sum_{j=3}^{n-1} n_1 n_j (j^2 - 1) + \sum_{j=3}^{n-1} n_2 n_j (j^2 - 4) \\ &\geq 3n_1n_2 + 8n_1 \sum_{j=3}^{n-1} n_j + 5n_2 \sum_{j=3}^{n-1} n_j \\ &= 8nn_1 + 5nn_2 - 8n_1^2 - 5n_2^2 - 10n_1n_2\,. \end{aligned}$$

Certainly, the equation $irr_{t,2}(G) = 8nn_1 + 5nn_2 - 8n_1^2 - 5n_2^2 - 10n_1n_2$ holds if and only if G is either $(1, 2, 3)$-tridegreed graph or $(1, 2)$-bidegreed graph. □

Next, we solve the problem of finding graphs with the first eight minimum $irr_{o,2}$ values from the class of all n-vertex trees for $n \geq 12$. By direct computations, we find that the first eight smallest $irr_{o,2}$ values for the n-vertex trees are 6, 24, 32, 40, 42, 48, 50 and 56 for $n \geq 12$. In what follows, we find all those graphs from the class of all n-vertex trees which satisfy the inequality

$$irr_{o,2}(T) \leq 56\,. \tag{5}$$

The following known result brings us one step closer to the solution of the above-mentioned extremal problem concerning $irr_{o,2}$.

Proposition 2.2. [19] *If T is a tree with maximum degree Δ then $irr_{o,2}(T) \geq \Delta(\Delta^2 - 1)$ with equality if and only if T is isomorphic to either a path or a tree containing only one vertex of degree greater than 2.*

Due to Proposition 2.2, in order to find all the n-vertex trees satisfying Inequality (5), it is enough to consider only those trees which have the maximum degree at most 3.

Lemma 2.1. [19] *Let uv be an edge of a graph G satisfying one of the following conditions*

1. *$deg(u) = 1$ and $deg(v) \geq 2$;*
2. *at least one of the vertices u, v has degree 2.*

If G' is the graph obtained from G by inserting a new vertex $x \notin V(G)$ of degree 2 on the edge uv, then $irr_{o,2}(G') = irr_{o,2}(G)$.

Let $P := v_0v_1 \cdots v_r$ be a path in a graph G. The path P is called a pendant path if $deg(v_0) \geq 3$, $deg(v_r) = 1$ and $deg(v_1) = deg(v_2) = \ldots = deg(v_{r-1}) = 2$. While, the path P is called an internal path if $deg(v_0), deg(v_l) \geq 3$ and $deg(v_1) = deg(v_2) = \ldots = deg(v_{r-1}) = 2$. An edge incident to a pendant vertex is called a pendant edge.

Corollary 2.4. *Let G be an n-vertex nonregular connected graph different from the path graph P_n. Let $G^\star$ be the graph obtained from G by replacing every pendant path of length greater than 1 with a pendant edge and every internal path of length at least 3 by an internal path of length 2. Then $irr_{o,2}(G) \geq 8n_1(G)$ with the equality if and only if $G^\star$ is a $(1,3)$-bidegreed graph.*

Proof. Suppose that $M = \{uv \in E(G^\star) : \ v \text{ is a pendant vertex of } G^\star\}$. By Lemma 2.1, we have $irr_{o,2}(G) = irr_{o,2}(G^\star)$ and so

$$\begin{aligned} irr_{o,2}(G) &= \sum_{uv \in M} |deg_{G^\star}(u)^2 - 1| + \sum_{uv \in E(G^\star) \setminus M} |deg_{G^\star}(u)^2 - deg_{G^\star}(v)^2| \\ &\geq \sum_{uv \in M} |deg_{G^\star}(u)^2 - 1| \geq 8n_1(G), \end{aligned}$$

with the equality if and only if $G^\star$ is a $(1,3)$-bidegreed graph. □

Lemma 2.2. [10] *If T is a tree of order n with $n_1 \leq 7$, then $n_3 \leq 5$.*

Lemma 2.3. [19] *Let uv be an edge of a graph G satisfying $deg(u) = deg(v) = 3$. If G' is the graph obtained from G by inserting a new vertex $x \notin V(G)$ of degree 2 on the edge uv, then $irr_{o,2}(G') = irr_{o,2}(G) + 10$.*

Table 1. All the classes of n-vertex trees satisfying $\Delta = 2$ or 3, $1 \leq n_3 \leq 5$ and $irr_{o,2} \leq 56$.

Class	n_3	n_2	n_1	$m_{3,3}$	n	$irr_{o,2}$
$\mathbb{A}_1$	0	$n-2$	2	0	$n \geq 3$	6
$\mathbb{A}_2$	1	$n-4$	3	0	$n \geq 4$	24
$\mathbb{A}_3$	2	$n-6$	4	1	$n \geq 6$	32
$\mathbb{A}_4$	3	$n-8$	5	2	$n \geq 8$	40
$\mathbb{A}_5$	2	$n-6$	4	0	$n \geq 7$	42
$\mathbb{A}_6$	4	$n-10$	6	3	$n \geq 10$	48
$\mathbb{A}_7$	3	$n-8$	5	1	$n \geq 9$	50
$\mathbb{A}_8$	5	$n-12$	7	4	$n \geq 12$	56

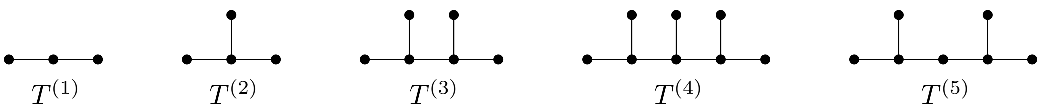

Figure 1: The trees $T^{(1)}, T^{(2)}, \cdots, T^{(5)}$.

Theorem 2.5. *For $n \geq 12$ and $i = 1, 2, \cdots, 8$, if $T_i \in \mathbb{A}_i$ and if T is an n-vertex tree different from the trees T_1, T_2, $\cdots$, T_8, then*

$$irr_{o,2}(T) > irr_{o,2}(T_{j+1}) > irr_{o,2}(T_j)\,,$$

for $j = 1, 2, \cdots, 7$, where the classes $\mathbb{A}_i$'s are defined in Table 1 (for $i = 1, 2, 3, 4, 5$, we note that $T_i \in \mathbb{A}_i$ is isomorphic to either the tree $T^{(i)}$, depicted in Figure 1, or some subdivision of $T^{(i)}$).

Proof. If $\Delta \geq 4$ then from Proposition 2.2, it follows that $irr_{o,2}(T) > 56$. If $n_3 \geq 6$ then by Lemma 2.2, we have $n_1 \geq 8$ and hence Corollary 2.4 ensures that $irr_{o,2}(T) > 56$. Bearing in mind Lemma 2.3, we find all the n-vertex trees satisfying $\Delta = 2$ or 3, $1 \leq n_3 \leq 5$ and $irr_{o,2} \leq 56$; see Table 1. □

Remark 2.1. *With the notations described in Theorem 2.5, the following statements hold.*

1. *If* $n = 10$ *or* 11*, then* $irr_{o,2}(T) > irr_{o,2}(T_{j+1}) > irr_{o,2}(T_j)$*, for* $j = 1, 2, \cdots, 6$.

2. *If* $n = 9$*, then* $irr_{o,2}(T) > irr_{o,2}(T_7) > irr_{o,2}(T_5) > irr_{o,2}(T_4) > irr_{o,2}(T_3) > irr_{o,2}(T_2) > irr_{o,2}(T_1)$.

3. *If* $n = 8$*, then* $irr_{o,2}(T) > irr_{o,2}(T_5) > irr_{o,2}(T_4) > irr_{o,2}(T_3) > irr_{o,2}(T_2) > irr_{o,2}(T_1)$.

4. *If* $n = 7$*, then* $irr_{o,2}(T) > irr_{o,2}(T_5) > irr_{o,2}(T_3) > irr_{o,2}(T_2) > irr_{o,2}(T_1)$.

5. *If* $n = 6$*, then* $irr_{o,2}(T) > irr_{o,2}(T_3) > irr_{o,2}(T_2) > irr_{o,2}(T_1)$.

6. *If* $n = 5$ *or* 4*, then* $irr_{o,2}(T) > irr_{o,2}(T_2) > irr_{o,2}(T_1)$.

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