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On the complementary distance energy of join of certain graphs

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Abstract

The complementary distance matrix of a graph G is defined as $CD(G) = [c_{ij}]$, in which $c_{ij} = 1 + D - d_{ij}$ if $i \neq j$ and $c_{ij} = 0$ if $i = j$, where D is the diameter of G and d_{ij} is the distance between the vertices v_i and v_j in G. The complementary distance energy $CDE(G)$ of G is defined as the sum of the absolute values of the eigenvalues of complementary distance matrix of G. Two graphs G_1 and G_2 are said to be CD-equienergetic if $CDE(G_1) = RCDE(G_2)$. In this paper, we obtain the CD-polynomial and CD-energy of the join of regular graphs of diameter at most two. We use these results to show that there exists at least one pair of CD-non-cospectral, CD-equienergetic graphs on n vertices, for every $n \geq 6$.

Keywords: complementary distance eigenvalues; complementary distance energy; equienergetic graphs.

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1. Introduction

Let G be a simple connected graph with n vertices and m edges. Let the vertex set of G be $V(G) = \{v_1, v_2, \ldots, v_n\}$. The *distance* between the vertices v_i and v_j, denoted by $d_{ij} = d(v_i, v_j)$ is the length of shortest path joining them. The *diameter* of a graph G, denoted by $diam(G)$, is the maximum distance between any pair of vertices of G [1].

The *complementary distance* between the vertices v_i and v_j is defined as $c_{ij} = 1 + D - d_{ij}$, where $D = diam(G)$ and d_{ij} is the distance between the vertices v_i and v_j in G. The *complementary distance (CD) matrix* [6] of a graph G is an $n \times n$ real symmetric matrix $CD(G) = [c_{ij}]$, where

$$c_{ij} = \begin{cases} 1 + D - d_{ij}, & \text{if } i \neq j \\ 0, & \text{otherwise.} \end{cases}$$

The complementary distance matrix gives a quantitative structure-property relationship (QSPR) models in chemistry. The structural descriptors computed with the CD-matrix are used to develop structure-property models for the boiling temperature, molar heat capacity, standard Gibbs energy of formation, vaporization enthalpy, refractive index and density of alkanes [6, 7].

The *characteristic polynomial* of $CD(G)$ is defined as $\psi(G : \mu) = \det(\mu I - CD(G))$, where I is the identity matrix of order n. The eigenvalues of the complementary distance matrix, denoted by $\mu_1, \mu_2, \ldots, \mu_n$, are said to be the *complementary distance eigenvalues* or CD*-eigenvalues* of G and their collection is called the CD*-spectra* of G. Two non-isomorphic graphs are said to be ***CD*-cospectral** if they have same CD-spectra. The *complementary distance energy* or ***CD*-energy** of a graph G denoted by $CDE(G)$ is defined as

$$CDE(G) = \sum_{i=1}^{n} |\mu_i| \,. \tag{1}$$

The Equation (1) is in full analogy to the *ordinary graph energy* [2], which is defined as the sum of the absolute values of the eigenvalues of the adjacency matrix of G. For details about the graph energy, one can refer the survey articles [3, 4, 9] and the books [5, 8].

Two graphs G_1 and G_2 are said to be *complementary distance equienergetic* or ***CD*-equienergetic** if $CDE(G_1) = CDE(G_2)$. Obviously, the CD-cospectral graphs are CD-equienergetic. Therefore, it is worthy to find CD-non-cospectral, CD-equienergetic graphs having the equal number of vertices. Distance and distance-like equienergetic graphs have been reported in [10–12]. In [13], the CD-energy of line graphs of certain regular graphs was obtained and thus the pairs of CD-equienergetic graphs were obtained. Here, we obtain the characteristic polynomial of the CD-matrix of the join of two regular graphs whose

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diameter is less than or equal to 2 and thereby we show that there exists at least one pair of CD-non-cospectral, CD-equienergetic graphs on n vertices, for every $n \geq 6$.

2. CD-spectra and CD-energy of join of graphs

Definition: The join of two graphs G_1 and G_2, denoted by $G_1 \nabla G_2$, is a graph obtained from G_1 and G_2 by joining each vertex of G_1 to all vertices of G_2 (for example, see Figure 1).

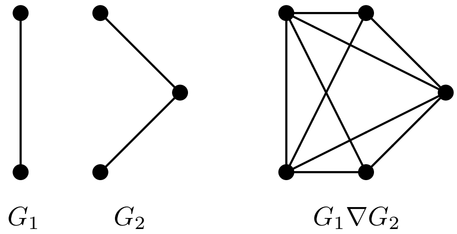

Figure 1: Two graphs G_1 and G_2, and their join.

Theorem 2.1. *Let G_i be an r_i-regular graph on n_i vertices and $diam(G_i) = 2$, $i = 1, 2$. Then the characteristic polynomial of the complementary distance matrix of $G_1 \nabla G_2$ is*

$$\psi(G_1 \nabla G_2 : \mu) = \frac{[(\mu - a_1)(\mu - a_2) - 4n_1n_2]}{(\mu - a_1)(\mu - a_2)} \psi(G_1 : \mu)\psi(G_2 : \mu), \tag{2}$$

where $a_1 = n_1 + r_1 - 1$ and $a_2 = n_2 + r_2 - 1$.

Proof.

$$\psi(G_1 \nabla G_2 : \mu) = \det(\mu I - CD(G_1 \nabla G_2)) = \begin{vmatrix} \mu I_{n_1} - CD(G_1) & (-2)J_{n_1 \times n_2} \\ (-2)J_{n_2 \times n_1} & \mu I_{n_2} - CD(G_2) \end{vmatrix}, \tag{3}$$

where J is the matrix whose all entries are equal to one and I is an identity matrix. The determinant given in the right hand side of (3) can be written as

$$\begin{vmatrix} \mu & -c_{12} & \cdots & -c_{1n_1} & -2 & -2 & \cdots & -2 \\ -c_{21} & \mu & \cdots & -c_{2n_1} & -2 & -2 & \cdots & -2 \\ \vdots & & \vdots & & & & \vdots & \\ -c_{n_11} & -c_{n_12} & \cdots & \mu & -2 & -2 & \cdots & -2 \\ -2 & -2 & \cdots & -2 & \mu & -c'_{12} & \cdots & -c'_{1n_2} \\ -2 & -2 & \cdots & -2 & -c'_{21} & \mu & \cdots & -c'_{2n_2} \\ \vdots & & \vdots & & & & \vdots & \\ -2 & -2 & \cdots & -2 & -c'_{n_21} & -c'_{n_22} & \cdots & \mu \end{vmatrix}, \tag{4}$$

where c_{ij} is the complementary distance between the vertices v_i and v_j in G_1 and c'_{ij} is the complementary distance between the vertices u_i and u_j in G_2. Since every G_i is an r_i-regular graph and $diam(G_i) = 2$ for $i = 1, 2$, every vertex of G_i is at distance one from r_i vertices and distance 2 from remaining $(n_i - 1 - r_i)$ vertices. Therefore,

$$\sum_{j=1}^{n_1} c_{ij} = n_1 + r_1 - 1 \qquad \text{for} \quad i = 1, 2, \ldots, n_1 \tag{5}$$

and

$$\sum_{j=1}^{n_2} c'_{ij} = n_2 + r_2 - 1 \qquad \text{for} \quad i = 1, 2, \ldots, n_2. \tag{6}$$

By subtracting the row $(n_1 + 1)$ from the rows $(n_1 + 2), (n_1 + 3), \ldots, (n_1 + n_2)$ of (4), we obtain (7)

$$\begin{vmatrix} \mu & -c_{12} & \cdots & -c_{1n_1} & -2 & -2 & \cdots & -2 \\ -c_{21} & \mu & \cdots & -c_{2n_1} & -2 & -2 & \cdots & -2 \\ \vdots & & \vdots & & & & \vdots & \\ -c_{n_11} & -c_{n_12} & \cdots & \mu & -2 & -2 & \cdots & -2 \\ -2 & -2 & \cdots & -2 & \mu & -c'_{12} & \cdots & -c'_{1n_2} \\ 0 & 0 & \cdots & 0 & -c'_{21} - \mu & \mu + c'_{12} & \cdots & -c'_{2n_2} + c'_{1n_2} \\ \vdots & & \vdots & & & & \vdots & \\ 0 & 0 & \cdots & 0 & -c'_{n_21} - \mu & -c'_{n_22} + c'_{12} & \cdots & \mu + c'_{1n_2} \end{vmatrix}. \tag{7}$$

By adding the columns $(n_1+2),(n_1+3),\ldots,(n_1+n_2)$ to the column (n_1+1) of (7) and using Equation (6), we get the determinant (8), where $a_2 = n_2 + r_2 - 1$.

$$\begin{vmatrix} \mu & -c_{12} & \cdots & -c_{1n_1} & -2n_2 & -2 & \cdots & -2 \\ -c_{21} & \mu & \cdots & -c_{2n_1} & -2n_2 & -2 & \cdots & -2 \\ \vdots & & \vdots & & & & \vdots & \\ -c_{n_11} & -c_{n_12} & \cdots & \mu & -2n_2 & -2 & \cdots & -2 \\ -2 & -2 & \cdots & -2 & \mu-a_2 & -c'_{12} & \cdots & -c'_{1n_2} \\ 0 & 0 & \cdots & 0 & 0 & \mu+c'_{12} & \cdots & -c'_{2n_2}+c'_{1n_2} \\ \vdots & & \vdots & & & & \vdots & \\ 0 & 0 & \cdots & 0 & 0 & -c'_{n_22}+c'_{12} & \cdots & \mu+c'_{1n_2} \end{vmatrix}. \tag{8}$$

Determinant (8) is equal to (9).

$$\begin{vmatrix} \mu & -c_{12} & \cdots & -c_{1n_1} & -2n_2 \\ -c_{21} & \mu & \cdots & -c_{2n_1} & -2n_2 \\ \vdots & & \vdots & & \\ -c_{n_11} & -c_{n_12} & \cdots & \mu & -2n_2 \\ -2 & -2 & \cdots & -2 & \mu-a_2 \end{vmatrix} |B|, \tag{9}$$

where

$$|B| = \begin{vmatrix} \mu+c'_{12} & -c'_{23}+c'_{13} & \cdots & -c'_{2n_2}+c'_{1n_2} \\ -c'_{32}+c'_{12} & \mu+c'_{13} & \cdots & -c'_{3n_2}+c'_{1n_2} \\ \vdots & & \vdots & \\ -c'_{n_22}+c'_{12} & -c'_{n_23}+c'_{13} & \cdots & \mu+c'_{1n_2} \end{vmatrix}. \tag{10}$$

The first determinant in (9) is of order (n_1+1). By subtracting the first row from the rows $2,3,\ldots,n_1$, in the first determinant of (9), we obtain (11).

$$\begin{vmatrix} \mu & -c_{12} & \cdots & -c_{1n_1} & -2n_2 \\ -c_{21}-\mu & \mu+c_{12} & \cdots & -c_{2n_1}+c_{1n_1} & 0 \\ \vdots & & \vdots & & \\ -c_{n_11}-\mu & -c_{n_12}+c_{12} & \cdots & \mu+c_{1n_1} & 0 \\ -2 & -2 & \cdots & -2 & \mu-a_2 \end{vmatrix} |B|\,. \tag{11}$$

Adding columns $2,3,\ldots,n_1$ to the first column of the first determinant in (11) and using Equation (5) we get (12), where $a_1 = n_1 + r_1 - 1$.

$$\begin{vmatrix} \mu-a_1 & -c_{12} & \cdots & -c_{1n_1} & -2n_2 \\ 0 & \mu+c_{12} & \cdots & -c_{2n_1}+c_{1n_1} & 0 \\ \vdots & & \vdots & & \\ 0 & -c_{n_12}+c_{12} & \cdots & \mu+c_{1n_1} & 0 \\ -2n_1 & -2 & \cdots & -2 & \mu-a_2 \end{vmatrix} |B|\,. \tag{12}$$

Expanding the first determinant of (12) along the first column gives (13).

$$\{(\mu-a_1)\,\Delta_1 - (-1)^{n_1}(2n_1)\Delta_2\}\,|B|, \tag{13}$$

where

$$\Delta_1 = \begin{vmatrix} \mu+c_{12} & -c_{23}+c_{13} & \cdots & -c_{2n_1}+c_{1n_1} & 0 \\ -c_{32}+c_{12} & \mu+c_{13} & \cdots & -c_{3n_1}+c_{1n_1} & 0 \\ \vdots & & \vdots & & \\ -c_{n_12}+c_{12} & -c_{n_13}+c_{13} & \cdots & \mu+c_{1n_1} & 0 \\ -2 & -2 & \cdots & -2 & \mu-a_2 \end{vmatrix}$$

and

$$\Delta_2 = \begin{vmatrix} -c_{12} & -c_{13} & \cdots & -c_{1n_1} & -2n_2 \\ \mu+c_{12} & -c_{23}+c_{13} & \cdots & -c_{2n_1}+c_{1n_1} & 0 \\ -c_{32}+c_{12} & \mu+c_{13} & \cdots & -c_{3n_1}+c_{1n_1} & 0 \\ \vdots & & \vdots & & \\ -c_{n_12}+c_{12} & -c_{n_13}+c_{13} & \cdots & \mu+c_{1n_1} & 0 \end{vmatrix}.$$

The expression (13) can be written as

$$\{(\mu-a_1)(\mu-a_2)\,|A| - (-1)^{n_1}(2n_1)(-1)^{1+n_1}(-2n_2)|A|\}\,|B| = \{(\mu-a_1)(\mu-a_2) - 4n_1n_2\}\,|A||B|, \tag{14}$$

where

$$|A| = \begin{vmatrix} \mu + c_{12} & -c_{23} + c_{13} & \cdots & -c_{2n_1} + c_{1n_1} \\ -c_{32} + c_{12} & \mu + c_{13} & \cdots & -c_{3n_1} + c_{1n_1} \\ \vdots & & \vdots & \\ -c_{n_1 2} + c_{12} & -c_{n_1 3} + c_{13} & \cdots & \mu + c_{1n_1} \end{vmatrix} . \tag{15}$$

The determinant (15) can be written as

$$|A| = \frac{1}{(\mu - a_1)} \begin{vmatrix} \mu - a_1 & -c_{12} & -c_{13} & \cdots & -c_{1n_1} \\ 0 & \mu + c_{12} & -c_{23} + c_{13} & \cdots & -c_{2n_1} + c_{1n_1} \\ 0 & -c_{32} + c_{12} & \mu + c_{13} & \cdots & -c_{3n_1} + c_{1n_1} \\ \vdots & & & \vdots & \\ 0 & -c_{n_1 2} + c_{12} & -c_{n_1 3} + c_{13} & \cdots & \mu + c_{1n_1} \end{vmatrix} . \tag{16}$$

From Equation (5), it follows that the sum of the i-th row in (16) is $\mu + c_{i1}$ for $i = 2, 3, \ldots, n_1$. Therefore, by subtracting the columns $2, 3, \ldots, n_1$ of (16) from the first column, we obtain (17).

$$|A| = \frac{1}{(\mu - a_1)} \begin{vmatrix} \mu & -c_{12} & -c_{13} & \cdots & -c_{1n_1} \\ -\mu - c_{21} & \mu + c_{12} & -c_{23} + c_{13} & \cdots & -c_{2n_1} + c_{1n_1} \\ -\mu - c_{31} & -c_{32} + c_{12} & \mu + c_{13} & \cdots & -c_{3n_1} + c_{1n_1} \\ \vdots & & & \vdots & \\ -\mu - c_{n_1 1} & -c_{n_1 2} + c_{12} & -c_{n_1 3} + c_{13} & \cdots & \mu + c_{1n_1} \end{vmatrix} . \tag{17}$$

Adding first row to the rows $2, 3, \ldots, n_1$ of (17), we get

$$|A| = \frac{1}{(\mu - a_1)} \begin{vmatrix} \mu & -c_{12} & -c_{13} & \cdots & -c_{1n_1} \\ -c_{21} & \mu & -c_{23} & \cdots & -c_{2n_1} \\ -c_{31} & -c_{32} & \mu & \cdots & -c_{3n_1} \\ \vdots & & & \vdots & \\ -c_{n_1 1} & -c_{n_1 2} & -c_{n_1 3} & \cdots & \mu \end{vmatrix} = \frac{1}{(\mu - a_1)} \psi(G_1 : \mu) . \tag{18}$$

Similarly, we can show that from (10) the following equation follows

$$|B| = \frac{1}{(\mu - a_2)} \psi(G_2 : \mu) . \tag{19}$$

Substituting (18) and (19) into (14) gives Equation (2). □

Theorem 2.2. *Let K_p be the complete graph on p vertices. Let G be an r-regular graph on n vertices and $diam(G) = 2$. Then the characteristic polynomial of the complementary distance matrix of $K_p \nabla G$ is*

$$\psi(K_p \nabla G : \mu) = \frac{[(\mu - 2p + 2)(\mu - b) - 4pn]}{(\mu - b)} (\mu + 2)^{p-1} \psi(G : \mu), \tag{20}$$

where $b = n + r - 1$.

Proof.

$$\psi(K_p \nabla G : \mu) = \det(\mu I - CD(K_p \nabla G)) = \begin{vmatrix} (\mu + 2)I_p - (2)J_{p \times p} & (-2)J_{p \times n} \\ (-2)J_{n \times p} & \mu I_n - CD(G) \end{vmatrix}, \tag{21}$$

where J is the matrix whose all entries are equal to one and I is an identity matrix. The determinant (21) can be written as

$$\begin{vmatrix} \mu & -2 & \cdots & -2 & -2 & -2 & \cdots & -2 \\ -2 & \mu & \cdots & -2 & -2 & -2 & \cdots & -2 \\ \vdots & & \vdots & & & & \vdots & \\ -2 & -2 & \cdots & \mu & -2 & -2 & \cdots & -2 \\ -2 & -2 & \cdots & -2 & \mu & -c_{12} & \cdots & -c_{1n} \\ -2 & -2 & \cdots & -2 & -c_{21} & \mu & \cdots & -c_{2n} \\ \vdots & & \vdots & & & & \vdots & \\ -2 & -2 & \cdots & -2 & -c_{n1} & -c_{n2} & \cdots & \mu \end{vmatrix}, \tag{22}$$

where c_{ij} is the complementary distance between the vertices v_i and v_j in G. Since G is an r-regular graph and $diam(G) = 2$, every vertex of G is at distance one from r vertices and distance 2 from remaining $(n - 1 - r)$ vertices. Therefore,

$$\sum_{j=1}^{n} c_{ij} = n + r - 1 \qquad \text{for} \quad i = 1, 2, \ldots, n. \tag{23}$$

By subtracting the row $(p+1)$ from the rows $(p+2),(p+3),\ldots,(p+n)$ of (22), we get

$$\begin{vmatrix} \mu & -2 & \cdots & -2 & -2 & -2 & \cdots & -2 \\ -2 & \mu & \cdots & -2 & -2 & -2 & \cdots & -2 \\ \vdots & & \vdots & & & & \vdots & \\ -2 & -2 & \cdots & \mu & -2 & -2 & \cdots & -2 \\ -2 & -2 & \cdots & -2 & \mu & -c_{12} & \cdots & -c_{1n} \\ 0 & 0 & \cdots & 0 & -c_{21}-\mu & \mu+c_{12} & \cdots & -c_{2n}+c_{1n} \\ \vdots & & \vdots & & & & \vdots & \\ 0 & 0 & \cdots & 0 & -c_{n1}-\mu & -c_{n2}+c_{12} & \cdots & \mu+c_{1n} \end{vmatrix}. \tag{24}$$

By adding the columns $(p+2),(p+3),\ldots,(p+n)$ to the column $(p+1)$ of (24) and using Equation (23) we arrive at the determinant (25), where $b=n+r-1$.

$$\begin{vmatrix} \mu & -2 & \cdots & -2 & -2n & -2 & \cdots & -2 \\ -2 & \mu & \cdots & -2 & -2n & -2 & \cdots & -2 \\ \vdots & & \vdots & & & & \vdots & \\ -2 & -2 & \cdots & \mu & -2n & -2 & \cdots & -2 \\ -2 & -2 & \cdots & -2 & \mu-b & -c_{12} & \cdots & -c_{1n} \\ 0 & 0 & \cdots & 0 & 0 & \mu+c_{12} & \cdots & -c_{2n}+c_{1n} \\ \vdots & & \vdots & & & & \vdots & \\ 0 & 0 & \cdots & 0 & 0 & -c_{n2}+c_{12} & \cdots & \mu+c_{1n} \end{vmatrix}, \tag{25}$$

which is equal to (26):

$$\begin{vmatrix} \mu & -2 & \cdots & -2 & -2n \\ -2 & \mu & \cdots & -2 & -2n \\ \vdots & & \vdots & & \\ -2 & -2 & \cdots & \mu & -2n \\ -2 & -2 & \cdots & -2 & \mu-b \end{vmatrix} |B|, \tag{26}$$

where

$$|B| = \begin{vmatrix} \mu+c_{12} & -c_{23}+c_{13} & \cdots & -c_{2n}+c_{1n} \\ -c_{32}+c_{12} & \mu+c_{13} & \cdots & -c_{3n}+c_{1n} \\ \vdots & & \vdots & \\ -c_{n2}+c_{12} & -c_{n3}+c_{13} & \cdots & \mu+c_{1n} \end{vmatrix}. \tag{27}$$

The first determinant in (26) is of order $(p+1)$. Subtracting first row from the rows $2,3,\ldots,p$, in the first determinant of (26), we get

$$\begin{vmatrix} \mu & -2 & \cdots & -2 & -2n \\ -2-\mu & \mu+2 & \cdots & 0 & 0 \\ \vdots & & \vdots & & \\ -2-\mu & 0 & \cdots & \mu+2 & 0 \\ -2 & -2 & \cdots & -2 & \mu-b \end{vmatrix} |B|\,. \tag{28}$$

Adding columns $2,3,\ldots,p$ to the first column of the first determinant in (28), we get

$$\begin{vmatrix} \mu-2(p-1) & -2 & \cdots & -2 & -2n \\ 0 & \mu+2 & \cdots & 0 & 0 \\ \vdots & & \vdots & & \\ 0 & 0 & \cdots & \mu+2 & 0 \\ -2p & -2 & \cdots & -2 & \mu-b \end{vmatrix} |B|\,. \tag{29}$$

Expanding it along the first column we get

$$\left\{(\mu-2(p-1))\,(\mu+2)^{p-1}(\mu-b)-4pn(\mu+2)^{p-1}\right\}|B| = \{(\mu-2p+2))\,(\mu-b)-4pn\}\,(\mu+2)^{p-1}|B|. \tag{30}$$

As done in Theorem 2.1, we can show that from (27), the following equation follows

$$|B| = \frac{1}{(\mu-b)}\,\psi(G:\mu)\,. \tag{31}$$

Substituting (31) into (30), we get Equation (20). □

Theorem 2.3. *Let G_i be an r_i-regular graph on n_i vertices and $diam(G_i) = 2$, where $i = 1, 2$. Then*

$$CDE(G_1 \nabla G_2) = CDE(G_1) + CDE(G_2) - (a_1 + a_2) + \sqrt{(a_1 - a_2)^2 + 16n_1n_2}\,,$$

where $a_1 = n_1 + r_1 - 1$ and $a_2 = n_2 + r_2 - 1$.

Proof. From Theorem 2.1, it follows that

$$\psi(G_1 \nabla G_2 : \mu) = \frac{[(\mu - a_1)(\mu - a_2) - 4n_1n_2]}{(\mu - a_1)(\mu - a_2)} \psi(G_1 : \mu)\psi(G_2 : \mu),$$

which gives that

$$(\mu - a_1)(\mu - a_2)\psi(G_1 \nabla G_2 : \mu) = [(\mu - a_1)(\mu - a_2) - 4n_1n_2]\, \psi(G_1 : \mu)\psi(G_2 : \mu).$$

Let $P_1(\mu) = (\mu - a_1)(\mu - a_2)\, \psi(G_1 \nabla G_2 : \mu)$ and $P_2(\mu) = [(\mu - a_1)(\mu - a_2) - 4n_1n_2]\; \psi(G_1 : \mu)\psi(G_2 : \mu)$. The roots of $P_1(\mu) = 0$ are a_1, a_2 and the CD-eigenvalues of $G_1 \nabla G_2$. Therefore, the sum of the absolute values of the roots of $P_1(\mu) = 0$ is

$$a_1 + a_2 + CDE(G_1 \nabla G_2). \tag{32}$$

The roots of $P_2(\mu) = 0$ are CD-eigenvalues of G_1 and G_2 and

$$\frac{1}{2}\left[(a_1 + a_2) \pm \sqrt{(a_1 + a_2)^2 - 4(a_1a_2 - 4n_1n_2)}\right].$$

Therefore, the sum of the absolute values of the roots of $P_2(\mu) = 0$ is

$$CDE(G_1) + CDE(G_2) + \left|\frac{1}{2}\left[(a_1 + a_2) + \sqrt{(a_1 + a_2)^2 - 4(a_1a_2 - 4n_1n_2)}\right]\right| + \left|\frac{1}{2}\left[(a_1 + a_2) - \sqrt{(a_1 + a_2)^2 - 4(a_1a_2 - 4n_1n_2)}\right]\right|. \tag{33}$$

Since $P_1(\mu) = P_2(\mu)$, equating (32) and (33), we get

$$\begin{aligned} CDE(G_1 \nabla G_2) = CDE(G_1) + CDE(G_2) - (a_1 + a_2) + \left|\frac{1}{2}\left[(a_1 + a_2) + \sqrt{(a_1 + a_2)^2 - 4(a_1a_2 - 4n_1n_2)}\right]\right| \\ + \left|\frac{1}{2}\left[(a_1 + a_2) - \sqrt{(a_1 + a_2)^2 - 4(a_1a_2 - 4n_1n_2)}\right]\right|. \end{aligned} \tag{34}$$

Since $r_1 \le n_1 - 1$ and $r_2 \le n_2 - 1$, it holds that

$$a_1a_2 = (n_1 + r_1 - 1)(n_2 + r_2 - 1) \le (2n_1 - 2)(2n_2 - 2) = 4(n_1 - 1)(n_2 - 1) < 4n_1n_2.$$

Therefore, Equation (34) reduces to

$$\begin{aligned} CDE(G_1 \nabla G_2) &= CDE(G_1) + CDE(G_2) - (a_1 + a_2) + \sqrt{(a_1 + a_2)^2 - 4(a_1a_2 - 4n_1n_2)} \\ &= CDE(G_1) + CDE(G_2) - (a_1 + a_2) + \sqrt{(a_1 - a_2)^2 + 16n_1n_2}\,. \end{aligned}$$

□

Corollary 2.1. *If H_1 and H_2 are CD-non-cospectral, CD-equienergetic, r-regular graphs on n vertices and $diam(H_i) = 2$, $i = 1, 2$, then for any regular graph G with $diam(G) = 2$, $CDE(H_1 \nabla G) = CDE(H_2 \nabla G)$.*

Theorem 2.4. *Let K_p be the complete graph on p vertices. Let G be an r-regular graph on n vertices and $diam(G) = 2$. Then*

$$CDE(K_p \nabla G) = CDE(G) + 2p - 2 - b + \sqrt{(2p - 2 - b)^2 + 16pn},$$

where $b = n + r - 1$.

Proof. From Theorem 2.2, it follows that

$$\psi(K_p \nabla G : \mu) = \frac{[(\mu - 2p + 2)(\mu - b) - 4pn]}{(\mu - b)} (\mu + 2)^{p-1}\psi(G : \mu),$$

which gives that

$$(\mu - b)\psi(K_p \nabla G : \mu) = [(\mu - 2p + 2)(\mu - b) - 4pn]\, (\mu + 2)^{p-1}\psi(G : \mu).$$

Let $Q_1(\mu) = (\mu - b)\, \psi(K_p \nabla G : \mu)$ and $Q_2(\mu) = [(\mu - 2p + 2)(\mu - b) - 4pn]\, (\mu + 2)^{p-1}\psi(G : \mu)$. The roots of $Q_1(\mu) = 0$ are b and the CD-eigenvalues of $K_p \nabla G$. Therefore, the sum of the absolute values of the roots of $Q_1(\mu) = 0$ is

$$b + CDE(K_p \nabla G). \tag{35}$$

The roots of $Q_2(\mu) = 0$ are CD-eigenvalues of G, -2 ($p-1$ times) and

$$\frac{1}{2}\left[(2p-2+b) \pm \sqrt{(2p-2+b)^2 - 4\left((2p-2)b - 4pn\right)}\right].$$

Therefore, the sum of the absolute values of the roots of $Q_2(\mu) = 0$ is

$$CDE(G) + 2p - 2 + \left|\frac{1}{2}\left[(2p-2+b) + \sqrt{(2p-2+b)^2 - 4\left((2p-2)b - 4pn\right)}\right]\right| + \left|\frac{1}{2}\left[(2p-2+b) - \sqrt{(2p-2+b)^2 - 4\left((2p-2)b - 4pn\right)}\right]\right|. \tag{36}$$

Since $Q_1(\mu) = Q_2(\mu)$, by equating (35) and (36), we get

$$CDE(K_p \nabla G) = CDE(G) + 2p - 2 - b + \left|\frac{1}{2}\left[(2p-2+b) + \sqrt{(2p-2+b)^2 - 4\left((2p-2)b - 4pn\right)}\right]\right| + \left|\frac{1}{2}\left[(2p-2+b) - \sqrt{(2p-2+b)^2 - 4\left((2p-2)b - 4pn\right)}\right]\right|. \tag{37}$$

Since $r \leq n-1$,

$$(2p-2)b = (2p-2)(n+r-1) \leq (2p-2)(2n-2) = 4(p-1)(n-1) < 4pn.$$

Therefore, Equation (37) reduces to

$$\begin{aligned} CDE(K_p \nabla G) &= CDE(G) + 2p - 2 - b + \sqrt{(2p-2+b)^2 - 4\left((2p-2)b - 4pn\right)} \\ &= CDE(G) + 2p - 2 - b + \sqrt{(2p-2-b)^2 + 16pn}. \end{aligned}$$

□

3. CD-equienergetic graphs

Theorem 3.1. *There exists atleast one pair of CD-non-cospectral, CD-equienergetic graphs on n vertices for every $n \geq 6$.*

Proof. Consider the graphs H_a and H_b as shown in Figure 2.

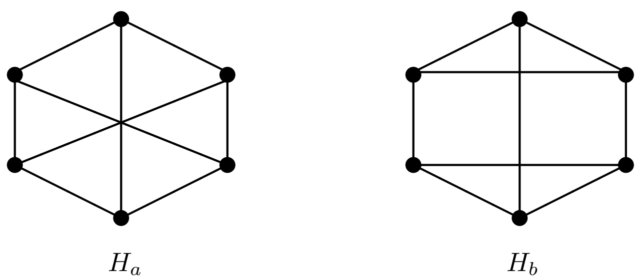

Figure 2: Graphs H_a and H_b.

We have,

$$\psi(H_a : \mu) = (\mu - 8)(\mu + 4)(\mu + 1)^4 \tag{38}$$

and

$$\psi(H_b : \mu) = \mu(\mu - 8)(\mu + 1)^2(\mu + 3)^2. \tag{39}$$

Both H_a and H_b are 3-regular graphs on 6 vertices. Also $diam(H_a) = 2 = diam(H_b)$ and $CDE(H_a) = CDE(H_b) = 16$. Let H be any r-regular graph on $p \geq 1$ vertices such that $diam(H) = 2$. Then, by Theorem 2.3,

$$CDE(H_a \nabla H) = CDE(H_b \nabla H) = CDE(H) + 9 - p - r + \sqrt{(9-p-r)^2 + 96p}.$$

Thus, $H_a \nabla H$ and $H_b \nabla H$ are CD-equienergetic graphs. By Equations (38) and (39), it follows that H_a and H_b are CD-non-cospectral, so from Theorem 2.1, it follows that $H_a \nabla H$ and $H_b \nabla H$ are CD-non-cospectral. Furthermore, $H_a \nabla H$ and $H_b \nabla H$ possesses equal number of vertices $n = 6 + p,\ p \geq 1$. Theorem holds also for $n = 6$ and it is directly verified from Equations (38) and (39). □

Using Theorem 2.4 we have the following result.

Theorem 3.2. *Let K_p be the complete graph on p vertices. If H_a and H_b are the graphs as shown in Figure 2, then*

$$CDE(K_p \nabla H_a) = CDE(K_p \nabla H_b) = 6 + 2p + \sqrt{(2p-10)^2 + 96p}.$$

4. Conclusion

The characteristic polynomial of the join of two regular graphs of diameter at most two is obtained and using it, the CD-energy of the join of two regular graphs of diameter at most two is expressed in terms of the CD-energies of the underline graphs. Using these results, the existence of CD-equienergetic graphs has been proved. By using Corollary 2.1, one can construct a pair of CD-non-cospectral, CD-equienergetic graphs. In particular, Theorems 3.1 and 3.2 reveals the construction of a pair of CD-non-cospectral, CD-equienergetic, n-vertex graphs for every $n \geq 6$. Further, it will be worthy to study this topic to find out other properties of CD-equienergetic graphs.

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