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# Staircase tableaux and an alternative matrix formula for steady state probabilities in the asymmetric exclusion process ($q = 1$)

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**Abstract**

We derive an alternative matrix formula for steady state probabilities in the asymmetric exclusion process where particles hop at equal rates inside a one-dimensional finite lattice. The result is derived using the combinatorial properties of staircase tableaux and alternative tableaux.

**Keywords:** asymmetric simple exclusion process; matrix ansatz; exclusion process; staircase tableau; alternative tableau.

**2010 Mathematics Subject Classification:** 05A10, 05A15, 60C05.

## 1. Introduction

The asymmetric simple exclusion process (ASEP) is a model of particle transport with applications in biology, physics and other fields. It belongs to a wide class of particle-hopping models called *exclusion processes*, where particles hop in a network subject to *hardcore exclusion*, or the property that at most one particle can occupy a site at a time. The particles can be protein molecules, quantum dots or even vehicles. For an account of the motivation behind the ASEP and details of its dynamics as a stochastic transport model, we refer the reader to [9, Chapter 4].

In this paper, we consider the ASEP on one-dimensional lattice of $N$ sites, where particles enter and exit from both ends of the lattice. We represent each state $\tau$ in the ASEP as a string in $\{0,1\}^N$, where the $t$-th letter $\tau(t)$ of $\tau$ equals 1 if the $t$-th site from the left of the lattice is occupied and 0 otherwise. Subject to hardcore exclusion, the transition probabilities are as follows: a particle enters from the left at probability $\alpha/(N+1)$, exits from the right at probability $\beta/(N+1)$, exits from the left at probability $\gamma/(N+1)$, enters from the right at probability $\delta/(N+1)$, hops to the left inside the lattice at probability $q/(N+1)$ and to the right at probability $1/(N+1)$. Moreover, if $P(\tau \to \tau')$ denotes the probability of transitioning from $\tau$ to $\tau'$, then $P(\tau \to \tau) = 1 - \sum P(\tau \to \tau')$, where the sum runs over all states $\tau'$ with transitions from $\tau$.

One of the key results in the study of the ASEP was derived by Derida *et al.* [4], who showed that given matrices $D$ and $E$, and column vectors $\mathbf{v}$ and $\mathbf{w}^T$ which satisfy the matrix ansatz

$$DE - qED = D + E, \quad \beta D\mathbf{v} - \delta E\mathbf{v} = \mathbf{v}, \quad \alpha\mathbf{w}E - \gamma\mathbf{w}D = \mathbf{w}\,, \tag{1}$$

the steady state probability of a state $\tau$ is given by

$$P(\tau;\alpha,\beta,\gamma,\delta,q) = \frac{\mathbf{w}\left(\prod_{t=1}^{N} \tau(t)D + (1-\tau(t))E\right)\mathbf{v}}{\mathbf{w}(E+D)^N\mathbf{v}}\,.$$

If $\tau$ is rewritten as a string in $\{E,D\}^N$ by replacing each $0$ with $E$ and each $1$ with $D$, then

$$P(\tau;\alpha,\beta,\gamma,\delta,q) = \frac{\mathbf{w}\tau\mathbf{v}}{\mathbf{w}(E+D)^N\mathbf{v}}\,. \tag{2}$$

Using (2) and by an abuse of notation, we can view $\tau \in \{E,D\}^N$ both as an ASEP state and a matrix product.

Denote the unnormalized numerator $\mathbf{w}\tau\mathbf{v}$ in (2) by $f(\tau;\alpha,\beta,\gamma,\delta,q)$, and the denominator, also called the *partition function*, by $Z_N(\alpha,\beta,\gamma,\delta,q)$. Note that $Z_N(\alpha,\beta,\gamma,\delta,q)$ is also equal to the sum of $f(\tau;\alpha,\beta,\gamma,\delta,q)$ over all the states $\tau$. When $q=1$, Corteel and Williams [2] and Uchiyama *et al.* [11] showed that $Z_N(\alpha,\beta,\gamma,\delta,1)$ has the following surprisingly simple form

$$Z_N(\alpha,\beta,\gamma,\delta,1) = \prod_{j=0}^{n-1} \left(\alpha+\beta+\gamma+\delta+j(\alpha+\gamma)(\beta+\delta)\right)\,.$$

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The study of the ASEP found its way into combinatorics as researchers obtain connections between computing the steady state probabilities and enumerating certain combinatorial objects. Shapiro and Zeilberger [10] expressed $P(\tau;1,1,0,0,0)$ in terms of the determinant of a matrix with binomial entries. Viennot [12] obtained a formula for $f(\tau;\alpha,\beta,0,0,q)$ as the generating function of alternative tableaux, while Corteel and Williams [3] expressed $f(\tau;\alpha,\beta,0,0,q)$ as the generating function of a related object called permutation tableaux. Corteel and Williams [2] also generalized their result to the case where all five parameters are arbitrary using staircase tableaux. Mandelshtam [6] obtained a compact formula for $f(\tau;\alpha,\beta,0,0,q)$ in terms of a combinatorial determinant. Connections with other combinatorial objects were discussed by Wood *et al.* [13]. More recently, results for generalizations of the ASEP involving multiple species of particles have also been obtained [5, 7, 8]. As to the single-species ASEP, while the probabilities have already been completely characterized via an enumerative formula, an explicit formula has not yet been found. Indeed, this is the case even for the three-parameter ASEP where $\delta=\gamma=0$.

In this paper, we revisit the five-parameter ASEP by focusing on the case when $q=1$. We write $P(\tau)$, $f(\tau)$ and $Z_N$ for simplicity, unless the value of $q$ is assumed to be arbitrary. The rest of the paper is organized as follows. In Section 2, we define the staircase tableau and the alternative tableau and recall some relevant results. We state our main result in Section 3, which is an expression for $f(\tau)$ as a product of linear combinations of known matrix solutions of the three-parameter ASEP. Some remarks are given in Section 4.

## 2. The ASEP, staircase tableaux and alternative tableaux

Let $0\le\alpha,\beta,\gamma,\delta\le q$ and let $N$ be the number of sites in the lattice. Corteel and Williams [2] derived an expression for $f(\tau)$ as the generating function of objects called *staircase tableau*, which is defined as follows.

**Definition 2.1.** A staircase tableau *of size $N$ is a Young diagram whose columns lengths from left to right are $n, n-1,\dots,2,1$, such that the boxes are labeled with $\alpha,\beta,\gamma$ or $\delta$ subject to the following conditions:*

- *no box along the diagonal is empty;*
- *all boxes in the same row and to the left of a $\beta$ or a $\delta$ are empty; and*
- *all boxes in the same column and above an $\alpha$ or a $\gamma$ are empty.*

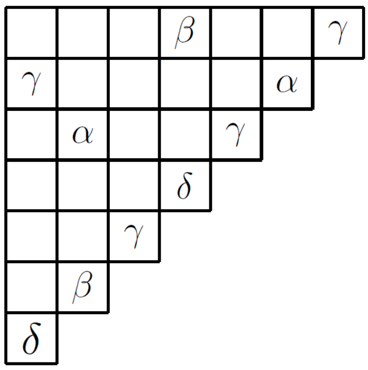

Figure 1: A staircase tableau.

Given a staircase tableau $T$, the *type* $(T)$ of $T$ is the word in $\{E,D\}^N$ obtained by reading the diagonal labels from northeast to southwest and writing a $D$ for each $\alpha$ or $\delta$ and an $E$ for each $\beta$ or $\gamma$. The *outline* $(T)$ of $T$ is defined similarly, except that we write a $D$ for each $\alpha$ or $\gamma$ and an $E$ for each $\beta$ or $\delta$. For $q=1$, we define the *weight* $(T)$ of $T$ as the product of the labels in all boxes. There are additional conditions when $q$ is arbitrary, for which the interested reader is referred to the paper [2]. Figure 1 gives an example of a staircase tableau of type $EDEDEED$, outline $DDDEDEE$, and weight $\alpha^2\beta^2\gamma^4\delta^2$.

For arbitrary $q$, Corteel and Williams [2] showed that

$$P(\tau;\alpha,\beta,\gamma,\delta,q)=\frac{\sum(T)}{Z_N(\alpha,\beta,\gamma,\delta,q)}\,, \tag{3}$$

where the sums runs over all staircase tableaux of type $\tau$. This result was proved by deriving an alternative matrix ansatz to (1).

We define a *condensed staircase tableau* as the collapsed tableau obtained from a staircase tableau by removing the columns (respectively, rows) that contain an $\alpha$ or $\gamma$ (respectively, $\beta$ or $\delta$) in the diagonal cell, and then labeling the underside border of the resulting tableau by the corresponding diagonal labels. Figure 2 gives the condensed staircase tableau from the staircase tableau in Figure 1.

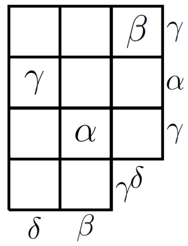


Figure 2: A condensed staircase tableau.

For a state $\tau \in \{E, D\}^N$, we denote by $B(\tau)$ the Ferrers board whose underside border is outlined by the lattice path obtained from the string $\tau$ by replacing $E$ and $D$ with a unit horizontal step and a unit vertical step, respectively. Notice that while the type of a staircase tableau gives the ASEP state to which it corresponds, its outline gives the shape of its condensed version.

Another object related to staircase tableaux are the alternative tableaux, which were used by Viennot [12] to compute the steady state probabilities. An example is shown in Figure 3.

**Definition 2.2.** *An* alternative tableau *is a Young diagram containing up and left arrows such that every cell pointed to by an arrow is empty.*

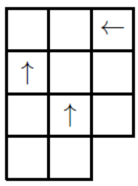

Figure 3: An alternative tableau.

Viennot showed that $P(\tau; \alpha, \beta, 0, 0, q)$ is proportional to the quantity

$$f^*(\tau; \alpha, \beta, 0, 0, q) = \sum_{\mathcal{A} \text{ of shape } B(\tau)} q^{\mathit{fcell}(\mathcal{A})} \alpha^{-\mathit{fcol}(\mathcal{A})} \beta^{-\mathit{frow}(\mathcal{A})}, \tag{4}$$

with statistics defined as follows: fcell($\mathcal{A}$) is the number of cells with no arrows pointing toward it, fcol($\mathcal{A}$) is the number of columns not containing an up arrow and frow($\mathcal{A}$) is the number of rows not containing a left arrow. For instance, in the alternative tableau shown in Figure 3, $\text{fcell} = 5, \text{fcol} = 1, \text{frow} = 3$.

Viennot's result can be restated as follows. Define the *weight* of an alternative tableau as $q$ raised to the number of cells not pointed to and not occupied by an arrow, and denote by $A_q^\tau(i, j)$ the total weight of all alternative tableaux of shape $B(\tau)$ for which $\text{frow} = i$ and $\text{fcol} = j$. Since $DE - qED = D + E$, each string in $\{E, D\}^N$ can be normally ordered; that is, written as the sum of monomials $E^i D^j$, by replacing every occurrence of $DE$ with $qED$, $D$ or $E$. In terms of placing arrows in a Ferrers board of shape $B(\tau)$ to form an alternative tableau, these replacements correspond to leaving a cell empty, placing an up arrow and placing a left arrow, respectively. We then have

$$\tau = \sum_{i,j} A_q^\tau(i, j) E^i D^j,$$

from which (4) follows using the other relations in the ansatz, namely $\beta D\mathbf{v} = \mathbf{v}$ and $\alpha\mathbf{w}E = \mathbf{w}$.

Given a staircase tableau containing only $\alpha$'s and $\beta$'s, we can obtain a unique alternative tableau by condensing the staircase tableau and replacing each cell label $\alpha$ by an up arrow and each $\beta$ by a left arrow. Observe that the quantity $f^*(\tau; \alpha, \beta, 0, 0, q)$ contains $\alpha$'s and $\beta$'s with negative exponents, which can be eliminated by multiplying $(\alpha\beta)^N$, i.e., $f(\tau; \alpha, \beta, 0, 0, q) = (\alpha\beta)^N f^*(\tau; \alpha, \beta, 0, 0, q)$.

## 3. Steady state probabilities as a matrix product

In this section, we derive an expression for the steady state probabilities as a matrix product which is slightly different from the form $P(\tau) = \mathbf{w}\tau\mathbf{v}$, but which also generalizes it when $q = 1$. This form presents the advantage that the matrices involved are solutions to the matrix ansatz of the three-parameter ASEP.

To state our main result, we will need the following. Denote by $a^\tau(i,j)$ the number of alternative tableaux of shape $B(\tau)$ containing $i$ left arrows and $j$ up arrows. In addition, for $\phi, \tau \in \{E,D\}^N$, define the *weight of* $\phi$ *with respect to* $\tau$ as the product

$$\mathrm{wt}_\tau(\phi) = \prod_{t=1}^{N} \left(\beta[\phi(t) = \tau(t) = E] + \gamma[\phi(t) = D, \tau(t) = E] + \alpha[\phi(t) = \tau(t) = D] + \delta[\phi(t) = E, \tau(t) = D]\right) .$$

The following lemma expresses the steady state probabilities in terms of the number of alternative tableaux.

**Lemma 3.1.** *Let* $0 \le \alpha, \beta, \gamma, \delta \le 1, q = 1$ *and* $\tau \in \{E,D\}^N$. *Then,*

$$P(\tau) = \frac{\sum_{\phi \in \{E,D\}^N} \sum_{i,j} \mathrm{wt}_\tau(\phi)(\beta+\delta)^i(\alpha+\gamma)^j a^\phi(i,j)}{Z_N} .$$

*Proof.* The basic idea is to collect all staircase tableaux of type $\tau$ having the same outline and express the sum of the weight of all such staircase tableau in terms of the number of alternative tableaux of that shape. Due to the restrictions on the placements of labels $\alpha, \beta, \gamma, \delta$ on a staircase tableau, we can think of a label of $\beta$ or $\delta$ as row-canceling and a label of either $\alpha$ or $\gamma$ as column-canceling, analogous to how left and up arrows in an alternative tableau cancel the cells they are pointing to.

First, we compute the weight contributed by diagonal labels. Pick a staircase tableau $T$ of type $\tau$. As previously described in Section 2, we obtain from $T$ a unique alternative tableau $T'$ via its corresponding condensed staircase tableau. This alternative tableau $T'$ has shape $B(\phi)$, where $\phi = (T)$. It can be verified that the product of the diagonal labels of $T$ is exactly $_\tau(\phi)$. For instance, if $\tau(t) = E$, then the $t$-th diagonal label of $T$ is either $\beta$ or $\gamma$. On the other hand, if $\phi(t) = D$, then the $t$-th column from the right of $T$ was canceled and therefore the $t$-th diagonal entry of $T$ is column-canceling and must therefore be $\gamma$. The rest of the conditions inside the Iverson brackets in the definition of $_\tau(\phi)$ can be verified similarly.

Now, we compute the weight contributed by all possible off-diagonal labels. Collect all staircase tableaux whose diagonal labels are identical to $T$ and whose number of row-canceling and column-cancelling labels are the same as $T$, say $i$ and $j$, respectively. These staircase tableaux have outline $\phi$ and correspond to the some alternative tableau of shape $B(\phi)$, and there are $a^\phi(i,j)$ ways to choose the position of the labels. Since there are two choices for row-canceling labels ($\beta$ or $\delta$) and two choices for column-canceling labels ($\alpha$ or $\gamma$), the total weight of all possible off-diagonal labels is $(\beta+\delta)^i(\alpha+\gamma)^j a^\tau(i,j)$. □

The following lemma translates Lemma 3.1 into a matrix ansatz.

**Lemma 3.2.** *Let* $0 \le \alpha, \beta, \gamma, \delta \le 1, q = 1$. *Let* $\tilde{E}, \tilde{D}$ *be matrices and* $\tilde{\mathbf{v}}, \tilde{\mathbf{w}}^T$ *be column vectors that satisfy the relations*

$$\tilde{\mathbf{w}}\tilde{\mathbf{v}} = 1, \quad \tilde{D}\tilde{E} = \tilde{E}\tilde{D} + (\alpha+\gamma)\tilde{D} + (\beta+\delta)\tilde{E}, \quad \tilde{D}\tilde{\mathbf{v}} = \tilde{\mathbf{v}}, \quad \textit{and} \quad \tilde{\mathbf{w}}\tilde{E} = \tilde{\mathbf{w}}.$$

*Then, for* $\tau \in \{E,D\}^N$ *we have*

$$P(\tau) = \frac{\sum_{\phi \in \{\tilde{E},\tilde{D}\}^N} \mathrm{wt}_\tau(\phi)\tilde{\mathbf{w}}\phi\tilde{\mathbf{v}}}{Z_N}, \tag{5}$$

*where* $\mathrm{wt}_\tau(\phi)$ *is defined with* $\phi$ *treated as a string in* $\{E,D\}^N$. *Equivalently,*

$$P(\tau) = \frac{\tilde{\mathbf{w}}\left(\prod_{t=1}^{N}(h^*(t)\tilde{E} + h'(t)\tilde{D})\right)\tilde{\mathbf{v}}}{Z_N}, \tag{6}$$

*where*

$$h^*(t) = \begin{cases} \beta, & \textit{if } \tau(t) = E \\ \delta, & \textit{if } \tau(t) = D \end{cases} \quad \textit{and} \quad h'(t) = \begin{cases} \alpha, & \textit{if } \tau(t) = D \\ \gamma, & \textit{if } \tau(t) = E\,. \end{cases}$$

*Proof.* Let $\phi \in \{\tilde{E},\tilde{D}\}^N$ and $\phi' \in \{E,D\}^N$ be $\phi$ considered as a word in $\{E,D\}^N$, i.e., by removing the tildes. Since $\tilde{D}, \tilde{E}$ satisfy $\tilde{D}\tilde{E} = \tilde{E}\tilde{D} + (\alpha+\gamma)\tilde{D} + (\beta+\delta)\tilde{E}$, $\phi$ can be normally ordered; that is, written as a sum of monomials $E^iD^j$ by replacing every leftmost $\tilde{D}\tilde{E}$ with $\tilde{E}\tilde{D}$, $(\alpha+\gamma)\tilde{D}$ or $(\beta+\delta)\tilde{E}$. In terms of alternative tableaux, replacing $\tilde{D}\tilde{E}$ with $\tilde{E}\tilde{D}$

corresponds to leaving a cell empty, with $(\alpha+\gamma)\tilde{D}$ to placing an up arrow with weight $(\alpha+\gamma)$, and with $(\beta+\delta)\tilde{E}$ to placing a left arrow with weight $(\beta+\delta)$. It follows that if $\phi$ has $\mathbf{e}$ $E$'s and $\mathbf{d}$ $D$'s, $\phi = \sum_{i,j}(\beta+\delta)^i(\alpha+\gamma)^j a^{\phi'}(i,j)E^{\mathbf{e}-i}D^{\mathbf{d}-j}$. Now, since $\mathrm{wt}_\tau(\phi) = \mathrm{wt}_\tau(\phi')$, and $\tilde{D}\tilde{\mathbf{v}} = \tilde{\mathbf{v}}$ and $\tilde{\mathbf{w}}\tilde{E} = \tilde{\mathbf{w}}$, we have

$$\begin{aligned}
\sum_{\phi\in\{\tilde{D},\tilde{E}\}^N} \mathrm{wt}_\tau(\phi)\tilde{\mathbf{w}}\phi\tilde{\mathbf{v}} &= \sum_{\phi\in\{\tilde{D},\tilde{E}\}^N} \mathrm{wt}_\tau(\phi)\sum_{i,j}(\beta+\delta)^i(\alpha+\gamma)^j a^{\phi'}(i,j)\tilde{\mathbf{w}}E^{\mathbf{e}-i}D^{\mathbf{d}-j}\tilde{\mathbf{v}} \\
&= \sum_{\phi\in\{\tilde{D},\tilde{E}\}^N} \mathrm{wt}_\tau(\phi)\sum_{i,j}(\beta+\delta)^i(\alpha+\gamma)^j a^{\phi'}(i,j) \\
&= \sum_{\phi\in\{D,E\}^N} \mathrm{wt}_\tau(\phi)\sum_{i,j}(\beta+\delta)^i(\alpha+\gamma)^j a^{\phi}(i,j) \\
&= Z_N P(\tau)
\end{aligned}$$

where the last equality follows from Lemma 3.1. This proves (5). Finally, (6) follows from (5) and the fact that

$$\prod_{t=1}^{N}(h^*(t)\tilde{E} + h'(t)\tilde{D}) = \sum_{\phi\in\{\tilde{E},\tilde{D}\}^N} \mathrm{wt}_\tau(\phi)\phi\,. \qquad \square$$

We now state our main result.

**Theorem 3.1.** *Let* $0 \le \alpha,\beta,\gamma,\delta \le 1, q = 1$. *In addition, let* $\hat{D},\hat{E}$ *be matrices and* $\hat{\mathbf{v}},\hat{\mathbf{w}}^T$ *be column vectors that satisfy the relations*

$$\hat{\mathbf{w}}\hat{\mathbf{v}} = 1, \quad \hat{D}\hat{E} = \hat{E}\hat{D} + \hat{D} + \hat{E}, \quad (\beta+\delta)\hat{D}\hat{\mathbf{v}} = \hat{\mathbf{v}}, \quad \textit{and} \ \ (\alpha+\gamma)\hat{\mathbf{w}}\hat{E} = \hat{\mathbf{w}}.$$

*Let* $h^*(t)$ *and* $h'(t)$ *be as in Lemma 3.2. Then, for* $\tau \in \{E,D\}^N$,

$$P(\tau) = \frac{\hat{\mathbf{w}}\left(\prod_{t=1}^{N}\left(h^*(t)\,(\alpha+\gamma)\,\hat{E} + h'(t)\,(\beta+\delta)\,\hat{D}\right)\right)\hat{\mathbf{v}}}{Z_N}.$$

*Proof.* Suppose that $\hat{D},\hat{E},\hat{\mathbf{v}},\hat{\mathbf{w}}$ satisfy the relations above. Define the matrices $\tilde{E} = (\alpha+\gamma)\,\hat{E}, \tilde{D} = (\beta+\delta)\,\hat{D}$ and the vectors $\tilde{\mathbf{v}} = \hat{\mathbf{v}}, \tilde{\mathbf{w}} = \hat{\mathbf{w}}$. Then

$$\begin{aligned}
\tilde{D}\tilde{E} &= (\alpha+\gamma)(\beta+\delta)\hat{D}\hat{E} \\
&= (\alpha+\gamma)(\beta+\delta)\left(\hat{E}\hat{D} + \hat{D} + \hat{E}\right) \\
&= \left((\alpha+\gamma)\hat{E}\right)\left((\beta+\delta)\hat{D}\right) + (\alpha+\gamma)\left((\beta+\delta)\hat{D}\right) + (\beta+\delta)\left((\alpha+\gamma)\hat{E}\right) \\
&= \tilde{E}\tilde{D} + (\alpha+\gamma)\tilde{D} + (\beta+\delta)\tilde{E}\,.
\end{aligned}$$

Furthermore, $\tilde{D}\tilde{\mathbf{v}} = (\beta+\delta)\hat{D}\hat{\mathbf{v}} = \hat{\mathbf{v}} = \tilde{\mathbf{v}}$ and $\tilde{\mathbf{w}}\tilde{E} = (\alpha+\gamma)\hat{\mathbf{v}}\hat{E} = \hat{\mathbf{w}} = \tilde{\mathbf{w}}$. Hence, $\tilde{D},\tilde{E},\tilde{\mathbf{v}},\tilde{\mathbf{w}}$ satisfy the relations in Lemma 3.2. $\square$

Suppose that $E,D,\mathbf{v},\mathbf{w}^T$ satisfy the matrix relations in the three-parameter ASEP, given by

$$\mathbf{w}\mathbf{v} = 1, \quad DE = qED + D + E, \quad \beta D\mathbf{v} = \mathbf{v}, \quad \text{and} \ \ \alpha\mathbf{w}E = \mathbf{w}. \tag{7}$$

Replacing the parameters $\alpha$ and $\beta$ above with $(\alpha+\gamma)$ and $(\beta+\delta)$, respectively, and setting $q = 1$ will then provide a solution to the matrices in Theorem 3.1, which gives the value of $f(\tau)$ in terms of products of linear combinations of these matrices. Solutions to (7) were derived by a number of authors [1,4,11]. We give here the solution due to Corteel and Williams [3]:

$$\begin{aligned}
D &= \begin{bmatrix} 0 & \beta^{-1} & 0 & 0 & \cdots \\ 0 & 0 & \beta^{-1} & 0 & \cdots \\ 0 & 0 & 0 & \beta^{-1} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix} \\
&= \left[\beta^{-1}[j = i+1]\right]_{\substack{1\le i<\infty \\ 1\le j<\infty}} \\
E &= \begin{bmatrix} \alpha^{-1} & 0 & 0 & 0 & \cdots \\ \alpha^{-1}\beta & 1+\alpha^{-1}q & 0 & 0 & \cdots \\ \alpha^{-1}\beta^2 & \beta(1+2\alpha^{-1}q) & 1+q+\alpha^{-1}q^2 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}
\end{aligned}$$

$$= \left[\beta^{i-j}\left(\alpha^{-1}q^{j-1}\binom{i-1}{j-1}+\sum_{r=0}^{j-2}\binom{i-j+r}{r}q^r\right)\right]_{\substack{1\le i<\infty\\1\le j<\infty}}$$

$$\mathbf{w}=(1,0,0,0,\ldots)\,,\qquad \mathbf{v}=(1,1,1,1,\ldots)^T$$

## 4. Some remarks

Derrida *et al.* [4] showed that matrix solutions to the ASEP are useful in computing the values of certain physical quantities. For instance, the density profile $\langle\tau(t)\rangle_N$, which is the average number of particles at site $t$, is given by

$$\langle\tau(t)\rangle_N=\frac{\sum_{\tau(t)=D}f(\tau)}{Z_N}=\frac{\mathbf{w}(E+D)^{t-1}D(E+D)^{n-1}\mathbf{v}}{Z_N}.$$

Let $\kappa=(\alpha+\gamma)(\beta+\delta)$ and suppose $q=1$. Using Theorem 3.1, $\langle\tau(t)\rangle_N$ takes the form

$$\langle\tau(t)\rangle_N=\frac{\hat{\mathbf{w}}(\kappa\hat{E}+\kappa\hat{D})^{t-1}\kappa\hat{D}(\kappa\hat{E}+\kappa\hat{D})^{N-1}\mathbf{v}}{Z_N}\,,$$

Here, $Z_N$ is the usual partition function in terms of $E, D, \mathbf{w}, \mathbf{v}$, which is also equal to

$$Z_N=\hat{\mathbf{w}}(\kappa\hat{E}+\kappa\hat{D})^N\hat{\mathbf{v}}\,.$$

Higher correlations $\langle\tau(s),\ldots,\tau(t)\rangle_N$ are computed analogously.

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