Discrete Mathematics Letters
www.dmlett.com

Discrete Math. Lett. **2** (2019) 18–25

Rainbow mean colorings of graphs

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(Received: 15 July 2019. Received in revised form: 26 August 2019. Accepted: 26 August 2019. Published online: 28 August 2019.)

Abstract

A mean coloring of a connected graph G of order 3 or more is an edge coloring c of G with positive integers where the average of the colors of the edges incident with each vertex v of G is an integer. This average is the chromatic mean of v. If distinct vertices have distinct chromatic means, then c is called a rainbow mean coloring of G. The maximum vertex color in a rainbow mean coloring c of G is the rainbow chromatic mean index of c and the rainbow chromatic mean index of the graph G is the minimum chromatic mean index among all rainbow mean colorings of G. It is shown that the rainbow chromatic mean index exists for every connected graph of order 3 or more. The rainbow chromatic mean index is determined for paths, cycles, complete graphs, and stars.

Keywords: chromatic mean; rainbow mean colorings; rainbow chromatic mean index.

2010 Mathematics Subject Classification: 05C07, 05C15, 05C78.

1. Introduction

It is graph theory folklore that in every nontrivial graph, there are always two vertices having the same degree. Indeed, this fact is listed (indirectly) among the 24 theorems in an article by David Wells [6], asking which of these 24 theorems is the most beautiful. A graph G was initially called *perfect* and later called *irregular* if the degrees of all vertices of G are distinct. Consequently, no nontrivial graph is perfect, that is, irregular.

Over the years, "irregular graphs" have been looked at in a variety of ways (see [1–3,5], for example). While no nontrivial graph is irregular, there are irregular multigraphs of each order $n \geq 3$. A multigraph M can be looked at as a labeled graph G_M where each edge uv of G_M is labeled with the positive integer equal to the number of parallel edges joining u and v in M. The degree of v in M is then the sum of the labels of the edges in G_M that are incident with v. Later each edge label was considered as an edge color and the sum of the labels incident with a vertex was referred to as its chromatic sum which became the color of the vertex.

In 1986, at the 250th Anniversary of Graph Theory Conference held at Indiana University-Purdue University Fort Wayne (now called Purdue University Fort Wayne), the concept of "irregularity strength" was introduced by Gary Chartrand, which is the smallest positive integer k for which an edge coloring from the set $[k] = \{1, 2, \ldots, k\}$ exists giving rise to vertex colors (chromatic sums), all of which are distinct (see [4]). Consequently, the problem was to determine the smallest positive integer k such that each edge of a graph can be colored with an element of $[k]$ in such a way that the vertex colors are distinct. This then results in a vertex coloring of the graph, often called a rainbow coloring since all vertex colors are distinct. Here, we consider edge colorings of graphs with positive integers such that each vertex color is the average of the colors of its incident edges and all vertex colors are distinct.

2. Rainbow mean index

An edge coloring c of a connected graph G of order 3 or more with positive integers is called a *mean coloring* of G if the *chromatic mean* $\mathrm{cm}(v)$ of each vertex v of G, defined by

$$\mathrm{cm}(v) = \frac{\sum_{e \in E_v} c(e)}{\deg v}, \text{ where } E_v \text{ is the set of edges incident with } v,$$

is an integer. If distinct vertices have distinct chromatic means, then the edge coloring c is called a *rainbow mean coloring* of G. The following result shows that for every connected graph of order 3 or more, such an edge coloring always exists.

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Theorem 2.1. *Every connected graph of order* 3 *or more has a rainbow mean coloring.*

Proof. Suppose that G is a connected graph with $E(G) = \{e_1, e_2, \ldots, e_m\}$ where $m \geq 2$. Thus, $\Delta(G) = \Delta \geq 2$. Let $k = 2\Delta$ and $t = \Delta! k^m$. Define the edge coloring $c : E(G) \to [t]$ by $c(e_i) = \Delta! k^i$ for $1 \leq i \leq m$. We show that the coloring c has the desired property. Assume, to the contrary, that there are two distinct vertices u and v of G such that $\mathrm{cm}(u) = \mathrm{cm}(v)$. Let $\deg u = r$ and $\deg v = s$, where $r \leq s$ say, and let $E_u = \{e_{i_1}, e_{i_2}, \ldots, e_{i_r}\}$ and $E_v = \{e_{j_1}, e_{j_2}, \ldots, e_{j_s}\}$ where $1 \leq i_1 < i_2 < \cdots < i_r \leq m$ and $1 \leq j_1 < j_2 < \cdots < j_s \leq m$. If $uv \notin E(G)$, then $E_u \cap E_v = \emptyset$; while if $uv \in E(G)$, then $E_u \cap E_v = \{uv\}$. Consequently,

$$\begin{aligned} \mathrm{cm}(u) &= \frac{\Delta!}{r}\left(k^{i_1} + k^{i_2} + \cdots + k^{i_r}\right) \\ \mathrm{cm}(v) &= \frac{\Delta!}{s}\left(k^{j_1} + k^{j_2} + \cdots + k^{j_s}\right), \end{aligned}$$

where both $\mathrm{cm}(u)$ and $\mathrm{cm}(v)$ are positive integers. We consider two cases, according to whether $r = s$ or $r < s$.

Case 1. $r = s$. Then $k^{i_1} + k^{i_2} + \cdots + k^{i_r} = k^{j_1} + k^{j_2} + \cdots + k^{j_r}$.

- First, suppose that $i_r \neq j_r$. We may assume that $i_r < j_r$. Let $p = j_r \geq 2$. Since $k = 2\Delta \geq 4$, it follows that $k^p > k + k^2 + \ldots + k^{p-1}$. However then,

$$k^{j_1} + k^{j_2} + \cdots + k^{j_r} \geq k^{j_r} = k^p > k + k^2 + \ldots + k^{p-1} \geq k^{i_1} + k^{i_2} + \cdots + k^{i_r},$$

  which is a contradiction.

- Next, suppose that $i_r = j_r$. Then $k^{i_1} + k^{i_2} + \cdots + k^{i_{r-1}} = k^{j_1} + k^{j_2} + \cdots + k^{j_{r-1}}$ and $i_{r-1} \neq j_{r-1}$. We can apply the argument above to produce a contradiction.

Case 2. $r < s$. Then $s\left[k^{i_1} + k^{i_2} + \cdots + k^{i_r}\right] = r\left[k^{j_1} + k^{j_2} + \cdots + k^{j_s}\right]$.

- First, suppose that $i_r < j_s$. Let $p = j_s \geq 2$. Since $1 > \frac{1}{k^{p-1}} + \frac{1}{k^{p-2}} + \cdots + \frac{1}{k}$, it follows that

$$2 > \tfrac{1}{k^{p-1}} + \tfrac{1}{k^{p-2}} + \cdots + \tfrac{1}{k} + 1 > \tfrac{1}{k^{p-2}} + \tfrac{1}{k^{p-3}} + \cdots + \tfrac{1}{k} + 1.$$

  Hence, $k = 2\Delta > \Delta\left(\frac{1}{k^{p-2}} + \frac{1}{k^{p-3}} + \cdots + 1\right)$. Because $\Delta \geq s/r$, it follows that

$$\begin{aligned} k^{j_1} + k^{j_2} + \cdots + k^{j_s} &\geq k^{j_s} = k^p = k(k^{p-1}) > \Delta\left(\frac{1}{k^{p-2}} + \frac{1}{k^{p-3}} + \cdots + 1\right)k^{p-1} \\ &= \Delta(k + k^2 + \cdots + k^{p-1}) \geq \frac{s}{r}(k + k^2 + \cdots + k^{p-1}) \\ &\geq \frac{s}{r}\left[k^{i_1} + k^{i_2} + \cdots + k^{i_r}\right], \end{aligned}$$

  which is a contradiction.

- Next, suppose that $i_r \geq j_s$. The argument in Case 1 shows that $k^{i_1} + k^{i_2} + \cdots + k^{i_r} > k^{j_1} + k^{j_2} + \cdots + k^{j_s}$. Since $r < s$, it follows that $1 \geq r/s$ and so

$$k^{i_1} + k^{i_2} + \cdots + k^{i_r} > k^{j_1} + k^{j_2} + \cdots + k^{j_s} > \tfrac{r}{s}\left[k^{j_1} + k^{j_2} + \cdots + k^{j_s}\right],$$

  which is a contradiction. □

For a rainbow mean coloring c of a graph G, the maximum vertex color is the *rainbow chromatic mean index* (or simply, the *rainbow mean index*) $\mathrm{rm}(c)$ of c. That is, $\mathrm{rm}(c) = \max\{\mathrm{cm}(v) : v \in V(G)\}$. The *rainbow chromatic mean index* (or the *rainbow mean index*) $\mathrm{rm}(G)$ of the graph G itself is defined as

$$\mathrm{rm}(G) = \min\{\mathrm{rm}(c) : \ c \text{ is a rainbow mean coloring of } G\}.$$

Consequently, if G is a connected graph of order $n \geq 3$, then $\mathrm{rm}(G) \geq n$.

For a mean coloring of a connected graph G, the *chromatic sum* $\mathrm{cs}(v)$ of a vertex v of G is the sum of the colors of the edges incident with v. Hence, $\mathrm{cs}(v) = \deg v \cdot \mathrm{cm}(v)$. The following is an elementary yet useful result.

Proposition 2.1. *If c is a mean coloring of a connected graph G, then*

$$\sum_{v \in V(G)} \mathrm{cs}(v) = 2 \sum_{e \in E(G)} c(e).$$

Proof. When the chromatic sums of the vertices of G are added, the color of each edge xy is counted twice, once in $\mathrm{cs}(x)$ and once in $\mathrm{cs}(y)$. □

Let G be a connected graph of order 3 or more with a mean coloring. A vertex v of G is called *chromatically even* if $\mathrm{cs}(v)$ is even and v is *chromatically odd* otherwise. The following is an immediate consequence of Proposition 2.1.

Proposition 2.2. *Let G be a connected graph with a mean coloring. Then G has an even number of chromatically odd vertices.*

Proof. By Proposition 2.1, the sum of the chromatic sums of all vertices of G is an even number. Therefore, there is an even number of chromatically odd vertices. □

A consequence of Proposition 2.2 is stated next.

Corollary 2.1. *Let G be a connected graph of order $n \geq 6$ where $n \equiv 2 \pmod 4$ such that all vertices of G are odd. Then* $\mathrm{rm}(G) \geq n + 1$.

Proof. Assume, to the contrary, that $\mathrm{rm}(G) = n$. Since $n \equiv 2 \pmod 4$ and $n \geq 6$, it follows that $n = 4k+2$ for some positive integer k. Hence, G has $2k + 1$ chromatically odd vertices. This contradicts Proposition 2.2. □

3. The rainbow mean index of paths and cycles

To illustrate the concepts we have described, we determine the rainbow mean index of each path P_n and cycle C_n of order $n \geq 3$, beginning with the path P_4, which we will see is a special case.

Proposition 3.1. $\mathrm{rm}(P_4) = 5$.

Proof. The edge coloring of P_4 in Figure 1 shows that $\mathrm{rm}(P_4) \leq 5$. Next, we show that $\mathrm{rm}(P_4) \geq 5$. Assume, to the contrary, that there is a rainbow mean coloring c of P_4 such that $\mathrm{rm}(c) = 4$. Let $P_4 = (v_1, v_2, v_3, v_4)$. Since $\{\mathrm{cm}(v_i) : 1 \leq i \leq 4\} = [4]$, no two edges can be colored the same. Consequently, since only one vertex is colored 1, this implies that $\mathrm{cm}(v_1) = 1$ or $\mathrm{cm}(v_4) = 1$, say the former. Therefore, $c(v_1v_2) = 1$. Hence, the edges of P_4 are colored with distinct odd integers. If some edge of P_4 is colored 7 or more, then some vertex of P_4 is colored 5 or more, which is impossible. Thus, $\{c(v_iv_{i+1}) : i = 1, 2, 3\} = \{1, 3, 5\}$ and so $\{c(v_2v_3), c(v_3v_4)\} = \{3, 5\}$. Whether $c(v_2v_3) = 3$ or $c(v_2v_3) = 5$, it follows that $\{\mathrm{cm}(v_i) : 1 \leq i \leq 4\} \neq [4]$, a contradiction. Thus, $\mathrm{rm}(P_4) \geq 5$ and so $\mathrm{rm}(P_4) = 5$. □

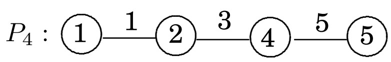

Figure 1: A rainbow mean coloring of P_4.

For all other paths P_n of order $n \geq 3$ and $n \neq 4$, the rainbow mean index of a path is its order. To show this, an appropriate rainbow mean coloring can be given.

Theorem 3.1. *For each integer $n \geq 3$ and $n \neq 4$,* $\mathrm{rm}(P_n) = n$.

The rainbow mean index of every cycle is determined next.

Theorem 3.2. *For each integer $n \geq 4$,*

$$\mathrm{rm}(C_n) = \begin{cases} n & \text{if } n \equiv 0, 1 \pmod 4 \\ n+1 & \text{if } n \equiv 2, 3 \pmod 4. \end{cases}$$

Proof. We consider two cases, according to whether $n \equiv 0, 1 \pmod 4$ or $n \equiv 2, 3 \pmod 4$.

Case 1. $n \equiv 0 \pmod 4$ *or* $n \equiv 1 \pmod 4$. In this case, it suffices to show that there is a rainbow mean coloring c of C_n such that $\mathrm{rm}(c) = n$. First, suppose that $n \equiv 0 \pmod 4$. Thus, $n = 4k$ for some positive integer k. Let C_{4k} be the cycle obtained from the paths $P = (u_1, u_2, \ldots, u_{2k})$ and $P' = (v_1, v_2, \ldots, v_{2k})$ by adding the two edges u_1v_1 and $u_{2k}v_{2k}$. The edge coloring $c : E(C_{4k}) \to [4k+1]$ is defined by

$$c(e) = \begin{cases} 1 & \text{if } e = u_1v_1 \\ 4k+1 & \text{if } e = u_{2k}v_{2k} \\ 2i+1 & \text{if } e = u_iu_{i+1} \text{ for } 1 \leq i \leq 2k-1 \\ 2i-1 & \text{if } e \in V(P') \text{ and } e \text{ is incident with } v_i \text{ where } i \text{ is odd and } 1 \leq i \leq 2k-1. \end{cases}$$

Note that there is exactly one edge $e = uv$ colored $n+1$ in C_n and $\{\mathrm{cm}(u), \mathrm{cm}(v)\} = \{n-1, n\}$. Then $\mathrm{cm}(u_i) = 2i$ for $1 \le i \le 2k$ and $cm(v_i) = 2i - 1$ for $1 \le i \le 2k$. Therefore, $\mathrm{rm}(C_{4k}) = 4k$ for each positive integer k.

Next, suppose that $n \equiv 1 \pmod 4$. Thus, $n = 4k+1$ where $k \in \mathbb{N}$ (the set of positive integers). Then C_n can be obtained by subdividing exactly one edge of C_{n-1}, where then $n-1 \equiv 0 \pmod 4$. A rainbow mean coloring c_n of C_n can be constructed from the rainbow mean coloring c_{n-1} of C_{n-1} described above by subdividing the edge $u_{2k}v_{2k}$ colored n by a new vertex w and coloring the two edges $u_{2k}w$ and wv_{2k} in C_n by n. Therefore, $\mathrm{rm}(C_{4k+1}) = 4k+1$ for each positive integer k.

Case 2. *$n \equiv 2 \pmod 4$ or $n \equiv 3 \pmod 4$.* Let $C = (v_1, v_2, \ldots, v_n, v_{n+1} = v_1)$ where $e_i = v_i v_{i+1}$ for $1 \le i \le n$. First, we show that $\mathrm{rm}(C_n) \ge n+1$. Assume, to the contrary, that $\mathrm{rm}(C_n) = n$. Then there is a rainbow mean coloring c of C_n such that $\{\mathrm{cm}(v) : v \in V(C_n)\} = [n]$. Since the color of some vertex of C_n is 1, the color of each edge incident with this vertex is also 1. This implies that $c(e)$ is odd for each $e \in E(C_n)$. Thus, $c(e_i) = 2a_i + 1$ for some nonnegative integer a_i where $1 \le i \le n$. First, suppose that $n \equiv 2 \pmod 4$. Then $n = 4k+2$ for some positive integer k. Hence,

$$2 \sum_{v \in V(C_n)} \mathrm{cm}(v) = 2\binom{4k+3}{2} = (4k+3)(4k+2) = 16k^2 + 20k + 6.$$

Hence, $2\sum_{v \in V(C_n)} \mathrm{cm}(v) \equiv 2 \pmod 4$. On the other hand,

$$\begin{aligned} 2 \sum_{v \in V(C_n)} \mathrm{cm}(v) &= 2\sum_{i=1}^{4k+2} c(e_i) = 2\sum_{i=1}^{4k+2} (2a_i + 1) = \sum_{i=1}^{4k+2} (4a_i + 2) \\ &= \left[\sum_{i=1}^{4k+2} 4a_i\right] + (8k+4) \equiv 0 \pmod 4, \end{aligned}$$

which is impossible. Next, suppose that $n \equiv 3 \pmod 4$. Thus, $n = 4k+3$ for some positive integer k. Then

$$2 \sum_{v \in V(C_n)} \mathrm{cm}(v) = 2\binom{4k+4}{2} = (4k+4)(4k+3) = 4(k+1)(4k+3).$$

Hence, $2\sum_{v \in V(C_n)} \mathrm{cm}(v) \equiv 0 \pmod 4$. On the other hand,

$$\begin{aligned} 2 \sum_{v \in V(C_n)} \mathrm{cm}(v) &= 2\sum_{i=1}^{4k+3} c(e_i) = 2\sum_{i=1}^{4k+3} (2a_i + 1) = \sum_{i=1}^{4k+3} (4a_i + 2) \\ &= \left[\sum_{i=1}^{4k+3} 4a_i\right] + (8k+6) \equiv 2 \pmod 4, \end{aligned}$$

which is impossible. Therefore, $\mathrm{rm}(C_n) \ge n+1$ if $n \equiv 2 \pmod 4$ or $n \equiv 3 \pmod 4$.

It remains to show that there exists a rainbow mean coloring c of C_n such that $\mathrm{rm}(c) = n+1$. First, suppose that $n = 4k+2$ for some positive integer k. Define $c : E(C_n) \to [n+1]$ by

$$c(e) = \begin{cases} i & \text{if } e \text{ is incident with } v_i,\ i \text{ is odd and } i \in [1, 2k-1] \\ i+2 & \text{if } e \text{ is incident with } v_i,\ i \text{ is odd and } i \in [2k+1, n-1]. \end{cases}$$

Consequently, the chromatic means of the vertices of C_n are given by

$$\mathrm{cm}(v_i) = \begin{cases} i & \text{if } i \text{ is odd and } i \in [1, 2k-1] \\ i+2 & \text{if } i \text{ is odd and } i \in [2k+1, n-1] \\ i & \text{if } i \text{ is even, } i \in [2, 2k-2] \text{ and } k \ge 2 \\ 2k+1 & \text{if } i = 2k \\ i+2 & \text{if } i \text{ is even and } i \in [2k+2, n-2] \\ 2k+2 & \text{if } i = n. \end{cases}$$

Next, suppose that $n \equiv 3 \pmod 4$ and so $n+1 \equiv 0 \pmod 4$. Then C_n can be obtained from C_{n+1} (colored as described above) by deleting a vertex v and joining the two neighbors u and w of v by the edge uw. A rainbow mean coloring c_n of C_n with $\mathrm{rm}(c_n) = n+1$ can be constructed from the rainbow mean coloring c_{n+1} of C_{n+1} with $\mathrm{rm}(c_{n+1}) = n+1$ in Case 1 by deleting the vertex v colored 1 and coloring the edge uw with 1. $\square$

4. The rainbow mean index of complete graphs

We now turn our attention to the complete graphs K_n of order $n \geq 3$. It is convenient here to consider the matrix representation of an edge-colored graph. Let G be a connected graph of order $n \geq 3$ with $V(G) = \{v_1, v_2, \ldots, v_n\}$ and let $c : E(G) \to \mathbb{N}$ be an edge coloring of G. The *matrix representation* M of G with the edge coloring c is the $n \times n$ matrix $[m_{i,j}]$ where

$$m_{i,j} = \begin{cases} c(v_iv_j) & \text{if } 1 \leq i \neq j \leq n \\ 0 & \text{if } 1 \leq i = j \leq n. \end{cases}$$

There are several elementary observations that can be made about the matrix representation M of a graph G of order n with an edge coloring c. First, all entries along the main diagonal of M are 0 since no vertex of G is adjacent to itself. Second, M is a symmetric matrix, that is, row i of M is identical to column i of M for every integer i with $1 \leq i \leq n$. Also, if we were to add the entries in row i (equivalently, in column i), then we obtain $\deg v_i \cdot \mathrm{cm}(v_i) = \mathrm{cs}(v_i)$ for $1 \leq i \leq n$. We now show that $\mathrm{rm}(K_n) = n$ for many integers $n \geq 4$.

Theorem 4.1. *For an integer $n \geq 4$ with $n \equiv 0, 1, 3 \pmod 4$, $\mathrm{rm}(K_n) = n$.*

Proof. Since $\mathrm{rm}(K_n) \geq n$, it suffices to show that there is a rainbow mean coloring of K_n having rainbow mean index n. We consider three cases.

Case 1. *$n \geq 4$ and $n \equiv 0 \pmod 4$.* Thus, $n = 4k$ for some positive integer k. In order to describe a rainbow mean coloring c_n of K_n with $\mathrm{rm}(c_n) = n$, we construct an $n \times n$ symmetric matrix M_n. First, we define, recursively, a sequence $B_1, B_2, \ldots, B_k$ of 4×4 symmetric matrices. For $a = n - 1$, let

$$B = \begin{bmatrix} 0 & a & a & 2a \\ a & 0 & 2a & a \\ a & 2a & 0 & a \\ 2a & a & a & 0 \end{bmatrix} \quad \text{and} \quad B_1 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & a+1 \\ 1 & 1 & 0 & 2a+1 \\ 1 & a+1 & 2a+1 & 0 \end{bmatrix}.$$

For $2 \leq i \leq k$, define $B_i = B_{i-1} + B = B_1 + (i-1)B$. Thus,

$$\begin{aligned} B_i &= B_1 + (i-1)B \\ &= \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & a+1 \\ 1 & 1 & 0 & 2a+1 \\ 1 & a+1 & 2a+1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & (i-1)a & (i-1)a & 2(i-1)a \\ (i-1)a & 0 & 2(i-1)a & (i-1)a \\ (i-1)a & 2(i-1)a & 0 & (i-1)a \\ 2(i-1)a & (i-1)a & (i-1)a & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & (i-1)a+1 & (i-1)a+1 & 2(i-1)a+1 \\ (i-1)a+1 & 0 & 2(i-1)a+1 & ia+1 \\ (i-1)a+1 & 2(i-1)a+1 & 0 & (i+1)a+1 \\ 2(i-1)a+1 & ia+1 & (i+1)a+1 & 0 \end{bmatrix}. \end{aligned}$$

To describe the $n \times n$ matrix M_n, we begin with a $k \times k$ matrix $A = [a_{i,j}]$ and then replace the entry $a_{i,i}$ on the main diagonal of A by the 4×4 matrix B_i for $1 \leq i \leq k$ and each entry off the main diagonal of A by the 4×4 matrix J, each of whose entries is 1. That is, $M_n = [M_{i,j}]$ is an $n \times n$ matrix, where $M_{i,j}$ is a 4×4 matrix such that

$$M_{i,j} = \begin{cases} B_i & \text{if } 1 \leq i = j \leq k \\ J & \text{if } 1 \leq i \neq j \leq k. \end{cases}$$

Thus, $M_4 = B_1$, $M_8 = \begin{bmatrix} B_1 & J \\ J & B_2 \end{bmatrix}$, and $M_{12} = \begin{bmatrix} B_1 & J & J \\ J & B_2 & J \\ J & J & B_3 \end{bmatrix}$. If we were to add the entries in row i (or in column i) in M_n, then we obtain ia for $1 \leq i \leq n$. That is, if $M_n = [m_{i,j}]$, then

$$\textstyle\sum_{j=1}^{n} m_{i,j} = ia = i(n-1) \text{ for } 1 \leq i \leq n. \tag{1}$$

We now define an edge coloring $c : E(K_n) \to \mathbb{N}$ by $c(v_iv_j) = m_{i,j}$ for each pair i, j of integers with $1 \leq i \leq j \leq n$ and $i \neq j$. Since $\mathrm{cm}(v_i) = \frac{1}{n-1}\sum_{j=1}^{n} m_{i,j} = i$ for $1 \leq i \leq n$ by (1), it follows that c is a rainbow mean coloring of K_n with $\mathrm{rm}(c) = n$. For example,

$$M_4 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 4 \\ 1 & 1 & 0 & 7 \\ 1 & 4 & 7 & 0 \end{bmatrix} \quad \text{and } M_8 = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 8 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 15 & 1 & 1 & 1 & 1 \\ 1 & 8 & 15 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 8 & 8 & 15 \\ 1 & 1 & 1 & 1 & 8 & 0 & 15 & 15 \\ 1 & 1 & 1 & 1 & 8 & 15 & 0 & 22 \\ 1 & 1 & 1 & 1 & 15 & 15 & 22 & 0 \end{bmatrix}.$$

The matrices M_4 and M_8 give rise to rainbow mean colorings of K_4 and K_8 as shown in Figure 2, respectively, where each edge drawn with a thin line is colored 1.

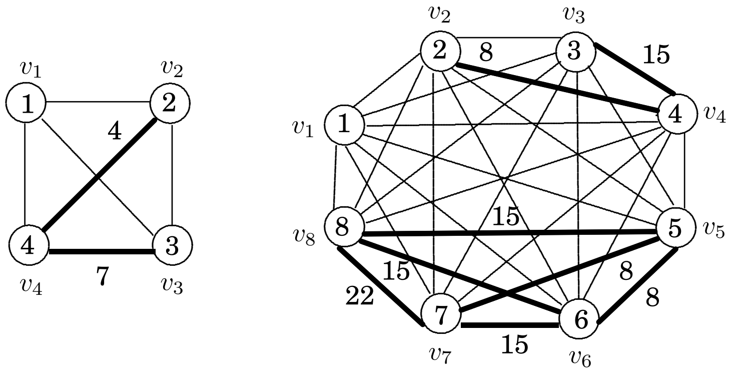

Figure 2: Rainbow mean colorings of K_4 and K_8.

Case 2. *$n \geq 5$ and $n \equiv 1 \pmod 4$. Then $n = 4k+1$ for some positive integer k.* First, we define, recursively, a sequence $B_1, B_2, \ldots, B_k$ of symmetric matrices, where B_1 is a 5×5 matrix and B_i is a 4×4 matrix for $2 \leq i \leq k$. For $a = n-1$, define

$$B_1 = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & a+1 & 1 & 1 \\ 1 & a+1 & 0 & 1 & a+1 \\ 1 & 1 & 1 & 0 & 3a+1 \\ 1 & 1 & a+1 & 3a+1 & 0 \end{bmatrix} \text{ and } B_2 = \begin{bmatrix} 0 & 2a+1 & a+1 & 2a+1 \\ 2a+1 & 0 & 2a+1 & 2a+1 \\ a+1 & 2a+1 & 0 & 4a+1 \\ 2a+1 & 2a+1 & 4a+1 & 0 \end{bmatrix}.$$

For $3 \leq i \leq k$, define $B_i = B_{i-1} + B = B_2 + (i-2)B$, where $B = \begin{bmatrix} 0 & a & a & 2a \\ a & 0 & 2a & a \\ a & 2a & 0 & a \\ 2a & a & a & 0 \end{bmatrix}$ was defined in Case 1. To describe the $n \times n$ matrix M_n, we begin with a $k \times k$ matrix $A = [a_{i,j}]$ and then replace the entry $a_{i,i}$ on the main diagonal of A by the matrix B_i for $1 \leq i \leq k$ and each entry off the main diagonal of A by the matrix J, each of whose entries is 1. Thus, $a_{1,1}$ is replaced by the 5×5 matrix B_1 and $a_{i,i}$ for $2 \leq i \leq k$ is replaced by the 4×4 matrix B_i. That is, $M_n = [M_{i,j}]$ is an $n \times n$ matrix, where

$$M_{i,j} = \begin{cases} B_i & \text{if } 1 \leq i = j \leq k \\ J & \text{if } 1 \leq i \neq j \leq k. \end{cases}$$

Thus, $M_5 = B_1$, $M_9 = \begin{bmatrix} B_1 & J \\ J & B_2 \end{bmatrix}$, and $M_{13} = \begin{bmatrix} B_1 & J & J \\ J & B_2 & J \\ J & J & B_3 \end{bmatrix}$.

Case 3. *$n \geq 7$ and $n \equiv 3 \pmod 4$.* Thus, $n = 4k+3$ for some positive integer k. Again, we construct an $n \times n$ symmetric matrix M_n. For $a = \frac{n-1}{2}$, let $C = \begin{bmatrix} 0 & 2a & 2a & 4a \\ 2a & 0 & 4a & 2a \\ 2a & 4a & 0 & 2a \\ 4a & 2a & 2a & 0 \end{bmatrix}$. First, we define, recursively, a sequence $C_1, C_2, \ldots, C_k$ of symmetric matrices, where C_1 is a 7×7 matrix and C_i is a 4×4 matrix for $2 \leq i \leq k$. Define

$$C_1 = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 2a+1 \\ 1 & 1 & 0 & 1 & 1 & 2a+1 & 2a+1 \\ 1 & 1 & 1 & 0 & 2a+1 & 2a+1 & 2a+1 \\ 1 & 1 & 1 & 2a+1 & 0 & 3a+1 & 3a+1 \\ 1 & 1 & 2a+1 & 2a+1 & 3a+1 & 0 & 3a+1 \\ 1 & 2a+1 & 2a+1 & 2a+1 & 3a+1 & 3a+1 & 0 \end{bmatrix} \text{ and } C_2 = \begin{bmatrix} 0 & 3a & 5a & 6a \\ 3a & 0 & 6a & 7a \\ 5a & 6a & 0 & 7a \\ 6a & 7a & 7a & 0 \end{bmatrix}.$$

For $3 \leq i \leq k$, define $C_i = C_{i-1} + C = C_2 + (i-2)C$. To describe the $n \times n$ matrix M_n, we begin with a $k \times k$ matrix $A = [a_{i,j}]$ and then replace the entry $a_{i,i}$ on the main diagonal of A by the matrix C_i for $1 \leq i \leq k$ and each entry off the main diagonal of A by the matrix J, each of whose entries is 1. That is, $M_n = [M_{i,j}]$ is an $n \times n$ matrix, where

$$M_{i,j} = \begin{cases} C_i & \text{if } 1 \leq i = j \leq k \\ J & \text{if } 1 \leq i \neq j \leq k. \end{cases}$$

Thus, $M_7 = C_1$ where $a = 3$, $M_{11} = \begin{bmatrix} C_1 & J \\ J & C_2 \end{bmatrix}$, where $a = 5$, and $M_{15} = \begin{bmatrix} C_1 & J & J \\ J & C_2 & J \\ J & J & C_3 \end{bmatrix}$ where $a = 7$. We now define a rainbow mean coloring $c : E(K_n) \to \mathbb{N}$ by $c(v_iv_j) = m_{i,j}$ for each pair i, j of integers with $1 \le i \le j \le n$ and $i \ne j$. Since $\mathrm{rm}(c) = n$, it follows that $\mathrm{rm}(K_n) = n$ for each integer $n \ge 7$ with $n \equiv 3 \pmod 4$. $\square$

The rainbow mean index of each remaining complete graph of order $n \ge 6$ is $n + 1$.

Theorem 4.2. *For an integer $n \ge 6$ with $n \equiv 2 \pmod 4$, $\mathrm{rm}(K_n) = n + 1$.*

Proof. By Corollary 2.1, it suffices to show that there is a rainbow mean coloring c_n of K_n with $\mathrm{rm}(c_n) = n + 1$. In order to do this, we construct an $n \times n$ symmetric matrix M_n by constructing a sequence $A_1, A_2, \ldots, A_k$ of symmetric matrices, where A_1 is a 6×6 matrix and A_i is a 4×4 matrix for $2 \le i \le k$. For $a = n - 1$, let $B = \begin{bmatrix} 0 & a & a & 2a \\ a & 0 & 2a & a \\ a & 2a & 0 & a \\ 2a & a & a & 0 \end{bmatrix}$. Define

$$A_1 = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & a+1 & 1 & 1 & 1 \\ 1 & a+1 & 0 & 1 & 1 & a+1 \\ 1 & 1 & 1 & 0 & a+1 & 2a+1 \\ 1 & 1 & 1 & a+1 & 0 & 3a+1 \\ 1 & 1 & a+1 & 2a+1 & 3a+1 & 0 \end{bmatrix} \text{ and } A_2 = \begin{bmatrix} 0 & a & 3a & 3a \\ a & 0 & 3a & 4a \\ 3a & 3a & 0 & 3a \\ 3a & 4a & 3a & 0 \end{bmatrix}.$$

For $3 \le i \le k$, define $A_i = A_{i-1} + B = A_2 + (i-2)B$.

To describe the $n \times n$ matrix M_n, we begin with a $k \times k$ matrix $A = [a_{i,j}]$ and then replace the entry $a_{i,i}$ on the main diagonal of A by the matrix A_i for $1 \le i \le k$ and each entry off the main diagonal of A by the matrix J, each of whose entries is 1. Thus, $a_{1,1}$ is replaced by the 6×6 matrix A_1 and $a_{i,i}$ for $2 \le i \le k$ is replaced by the 4×4 matrix A_i. That is, $M_n = [M_{i,j}]$ is an $n \times n$ matrix where

$$M_{i,j} = \begin{cases} A_i & \text{if } 1 \le i = j \le k \\ J & \text{if } 1 \le i \ne j \le k. \end{cases}$$

Thus, $M_6 = A_1$, $M_{10} = \begin{bmatrix} A_1 & J \\ J & A_2 \end{bmatrix}$ and $M_{14} = \begin{bmatrix} A_1 & J & J \\ J & A_2 & J \\ J & J & A_3 \end{bmatrix}$. We now define a rainbow mean coloring $c : E(K_n) \to \mathbb{N}$ by $c(v_iv_j) = m_{i,j}$ for each pair i, j of integers with $1 \le i \le j \le n$ and $i \ne j$. Since $\mathrm{rm}(c) = n+1$, it follows that $\mathrm{rm}(K_n) = n+1$ for each integer $n \ge 6$ with $n \equiv 2 \pmod 4$. $\square$

From Theorems 4.1 and 4.2, we then have the following result.

Corollary 4.1. *For an integer $n \ge 3$,*

$$\mathrm{rm}(K_n) = \begin{cases} n & \text{if } n \ge 4 \text{ and } n \equiv 0, 1, 3 \pmod 4 \\ n+1 & \text{if } n = 3 \text{ or } n \equiv 2 \pmod 4. \end{cases}$$

5. The rainbow mean index of stars

In a rainbow mean coloring of a connected graph G of order at least 3, each edge of G is colored with a positive integer in a manner so that each vertex is assigned a positive integer color that is the average of the colors of its incident edges and no two vertices of G have the same color. Therefore, as we saw, if the order of G is $n \ge 3$, then the number of vertex colors must therefore be at least n. With all the conditions required for a graph to have such an edge coloring, one might anticipate that for some graphs at least, the largest vertex color may exceed the order of the graph by a large degree. However, for each connected graph G of order $n \ge 3$ that we have considered thus far, we have seen that either $\mathrm{rm}(G) = n$ or $\mathrm{rm}(G) = n + 1$. While this observation may suggest a conjecture, the following result dealing with the stars $K_{1,n-1}$ of order $n \ge 3$ indicates that the value of $\mathrm{rm}(G)$ for a connected graph G of order $n \ge 3$ can be one of at least *three* integers rather than only one of two integers.

Theorem 5.1. *If G is a star of order $n \ge 3$, then*

$$\mathrm{rm}(G) = \begin{cases} n & \text{if } n \text{ is odd} \\ n+2 & \text{if } n \text{ is even.} \end{cases}$$

Proof. Let $G = K_{1,n-1}$ where $V(G) = \{v, v_1, v_2, \ldots, v_{n-1}\}$ and $\deg v = n-1$. First, suppose that n is odd. Thus, $n = 2t+1$ for some positive integer t. Define the coloring $c : E(G) \to [n]$ by $c(vv_i) = i$ for $1 \le i \le t$ and $c(vv_i) = i+1$ for $t+1 \le i \le 2t$. Since $\mathrm{cm}(v) = \frac{1}{2t}\left[\sum_{i=1}^{2t+1} i - (t+1)\right] = t+1$ and $\mathrm{cm}(v_i) = c(vv_i)$ for $1 \le i \le 2t$, it follows that c is a rainbow mean coloring with $\mathrm{rm}(c) = n$. Therefore, $\mathrm{rm}(G) = n$ if n is odd.

Next, suppose that $n \ge 4$ is even. Then $n = 2t$ for some integer $t \ge 2$. First, we show that there is a rainbow mean coloring c of G with $\mathrm{rm}(c) = n+2$. Define $c : E(G) \to \mathbb{N}$ such that $\{c(vv_i) : 1 \le i \le 2t-1\} = [2t+2] - \{t+1, t+2, 2t+1\}$. Since

$$\begin{aligned}\mathrm{cm}(v) &= \frac{1}{2t-1}\sum_{i=1}^{2t-1} c(vv_i) = \frac{1}{2t-1}\left[\binom{2t+3}{2} - (t+1) - (t+2) - (2t+1)\right] \\ &= \frac{1}{2t-1}\left[(2t+3)(t+1) - (4t+4)\right] = t+1\end{aligned}$$

and $\mathrm{cm}(v_i) = c(vv_i)$ for $1 \le i \le 2t-1$, it follows that c is a rainbow mean coloring of G with $\mathrm{rm}(c) = 2t+2$. Therefore, $\mathrm{rm}(G) \le n+2$.

It remains to show that $\mathrm{rm}(G) \ge n+2 = 2t+2$. Assume, to the contrary, that there is a rainbow mean coloring c of G such that $\mathrm{rm}(c) \in \{2t, 2t+1\}$. We consider two cases, according to whether $\mathrm{rm}(c) = 2t$ or $\mathrm{rm}(c) = 2t+1$.

Case 1. $\mathrm{rm}(c) = 2t$. If t is odd, then $n \equiv 2 \pmod 4$ and all vertices of G are odd. By Corollary 2.1, no such rainbow mean coloring c exists. We show that no such rainbow mean coloring c exists regardless of the parity of t. Then $\{\mathrm{cm}(u) : u \in V(G)\} = [2t]$. Since $\mathrm{cm}(v_i) = c(vv_i)$ for $1 \le i \le 2t-1$, it follows that $\{c(vv_i) : 1 \le i \le 2t-1\} = [2t] - \{a\}$ for some integer $a \in [2t]$. Thus,

$$\mathrm{cm}(v) = \frac{1}{2t-1}\left[\binom{2t+1}{2} - a\right] = \frac{1}{2t-1}[t(2t+1) - a] = \frac{1}{2t-1}(2t^2 + t - a).$$

If $a = 1$, then $\mathrm{cm}(v) = t+1$; while if $a = 2t$, then $\mathrm{cm}(v) = t$. In either case, $\mathrm{cm}(v) = \mathrm{cm}(v_i)$ for some integer i with $1 \le i \le 2t-1$, which is impossible. On the other hand, if $1 < a < 2t$, then $\mathrm{cm}(v)$ is not an integer, which is also impossible.

Case 2. $\mathrm{rm}(c) = 2t+1$. Then $\{\mathrm{cm}(u) : u \in V(G)\} \subseteq [2t+1]$. Since $\mathrm{cm}(v_i) = c(vv_i)$ for $1 \le i \le 2t-1$, it follows that $\{c(vv_i) : 1 \le i \le 2t-1\} = [2t+1] - \{a, b\}$ for some $a, b \in [2t+1]$ and $a \ne b$. Thus,

$$\begin{aligned}\mathrm{cm}(v) &= \frac{1}{2t-1}\left[\binom{2t+2}{2} - (a+b)\right] = \frac{1}{2t-1}[(t+1)(2t+1) - (a+b)] \\ &= \frac{1}{2t-1}[(2t^2+3t+1) - (a+b)].\end{aligned}$$

- ⋆ If $a = 1$ and $b = 2$, then $\mathrm{cm}(v) = t+2$;
- ⋆ If $a = 2t$ and $b = 2t+1$, then $\mathrm{cm}(v) = t$;
- ⋆ If $\mathrm{cm}(v) = t+1$, then $a + b = 2t+2$, where $1 \le a < t+1 < b \le 2t+1$.

In any of these situations, $\mathrm{cm}(v) = \mathrm{cm}(v_i)$ for some integer i with $1 \le i \le 2t-1$, which is impossible. For any other choice of a and b, it follows that $\mathrm{cm}(v)$ is not an integer, which is also impossible. □

We then close with the following conjecture.

Conjecture 5.1. *For every connected graph G of order $n \ge 3$,*

$$n \le \mathrm{rm}(G) \le n+2.$$

References

[1] G. Chartrand, Highly Irregular, In: R. Gera, S. Hedetniemi, C. Larson (Eds), *Graph Theory-Favorite Conjectures and Open Problems*, Springer, New York, 2016.
[2] G. Chartrand, C. Egan, P. Zhang, *How to Label a Graph,* Springer, New York, 2019.
[3] G. Chartrand, P. Erdős, O. R. Oellermann, How to define an irregular graph, *College Math. J.* **19** (1988) 36–42.
[4] G. Chartrand, M. S. Jacobson, J. Lehel, O. R. Oellermann, S. Ruiz, F. Saba, Irregular networks, *Congr. Numer.* **64** (1988) 197–210.
[5] G. Chartrand, P. Zhang, *Chromatic Graph Theory,* Second Edition, Chapman & Hall/CRC Press, Boca Raton, 2020.
[6] D. Wells, Which is the most beautiful? *Math. Intelligencer* **10** (1988) 30–31.