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Semi-polytope decomposition of a generalized permutohedron

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Abstract

In this short note, we show explicitly how to decompose a generalized permutohedron into semi-polytopes.

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1. Introduction

Given a polytope, assume we have disjoint open cells whose closures sum up to be the entire polytope. A question of naturally assigning each of the remaining points (possibly in multiple closures) to a cell has appeared in [1] for studying regular matroids and zonotopes and in [2] for studying h-vectors and Q-polytopes. In other words, we are trying to determine ownership of lattice points on boundaries of multiple polytopes. In this note, we study a more general case of doing the same for a ***Generalized permutohedron***, a polytope that can be obtained by deforming the usual permutohedron. We will show explicitly how to construct a semi-polytope decomposition of a trimmed generalized permutohedron.

2. Generalized permutohedron P_G and its fine mixed subdivision

Let $\Delta_{[n]} = \text{ConvexHull}(e_1, \dots, e_n)$ be the standard coordinate simplex in $\mathbb{R}^n$. For a subset $I \subset [n]$, let $\Delta_I = \text{ConvexHull}(e_i | i \in I)$ denote the face of $\Delta_{[n]}$. Let $G \subseteq K_{m,n}$ be a bipartite graph with no isolated vertices. Label the vertices of G by $1, \dots, m, \bar{1}, \dots, \bar{n}$ and call $1, \dots, m$ the ***left vertices*** and $\bar{1}, \dots, \bar{n}$ the right vertices. We identify the barred indices with usual non-barred cases when it is clear we are dealing with the right vertices. For example when we write $\Delta_{\{\bar{1},\bar{3}\}}$ we think of it as $\Delta_{\{1,3\}}$. We associate this graph with the collection $\mathcal{I}_G$ of subsets $I_1, \dots, I_m \subseteq [n]$ such that $j \in I_i$ if and only if $(i, \bar{j})$ is an edge of G. Let us define the polytope $P_G(y_1, \dots, y_m)$ as:

$$P_G(y_1, \dots, y_m) := y_1\Delta_{I_1} + \cdots + y_m\Delta_{I_m},$$

where y_i are nonnegative integers. This lies on a hyperplane

$$\sum_{i \in [n]} x_i = \sum_{j \in [m]} y_j.$$

An example of a coordinate simplex $\Delta_{[3]}$, a bipartite graph G and a generalized permutohedron $P_G(1, 2, 3)$ is given in Figure 1.

Definition 2.1 ([3], Definition 14.1). *Let d be the dimension of the Minkowski sum $P_1 + \cdots + P_m$. A* **Minkowski cell** *in this sum is a polytope $B_1 + \cdots + B_m$ of dimension d where B_i is the convex hull of some subset of vertices of P_i. A* **mixed subdivision** *of the sum is the decomposition into union of Minkowski cells such that intersection of any two cells is their common face. A mixed subdivision is* **fine** *if for all cells $B_1 + \cdots + B_m$, all B_i are simplices and $\sum dim B_i = d$.*

All mixed subdivisions in our note, unless otherwise stated, will be referring to fine mixed subdivisions. We will use the word ***cell*** to denote the Minkowski cells. Beware that our cells are all closed polytopes.

Fine Minkowski cells can be described by spanning trees of G. When we are looking at a fixed generalized permutohedron $P_G(y_1, \dots, y_m)$, we will use $\prod_J$ to denote $y_1\Delta_{J_1} + \cdots + y_m\Delta_{J_m}$ where $J = (J_1, \dots, J_m)$. We say that J is a tree if the associated bipartite graph is a tree.

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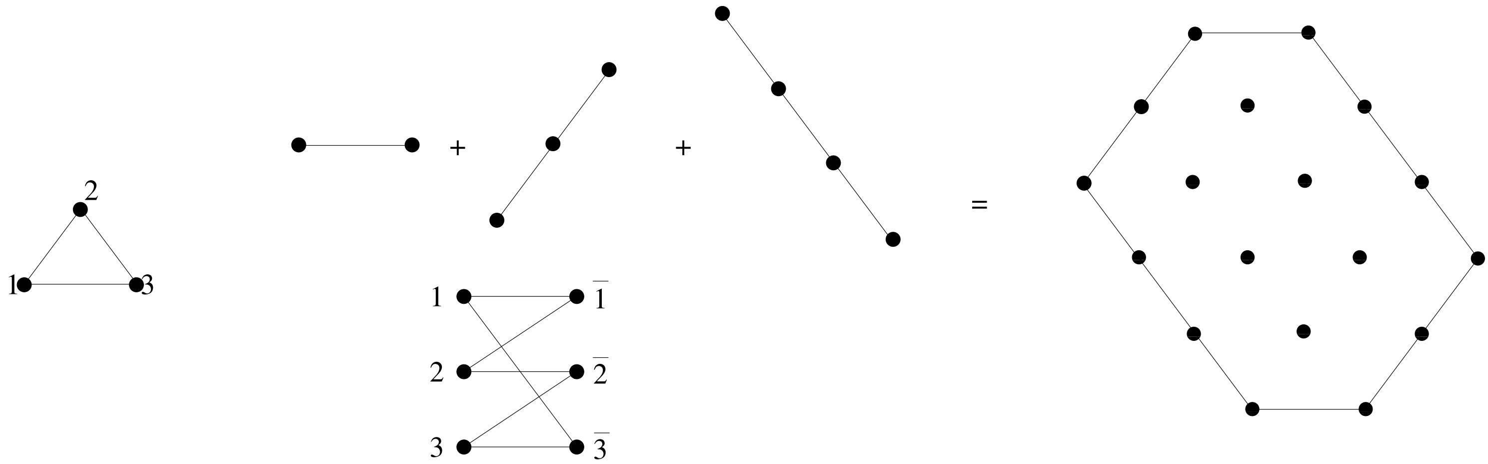

Figure 1: Example of a generalized permutohedron $P_G(1, 2, 3)$.

Lemma 2.1 ([3], Lemma 14.7)**.** *Each fine mixed cell in a mixed subdivision of $P_G(y_1, \cdots, y_m)$ has the form $\prod_T$ such that T is a spanning tree of G.*

An example of a fine mixed subdivision of the polytope considered in Figure 1 is given in Figure 2.

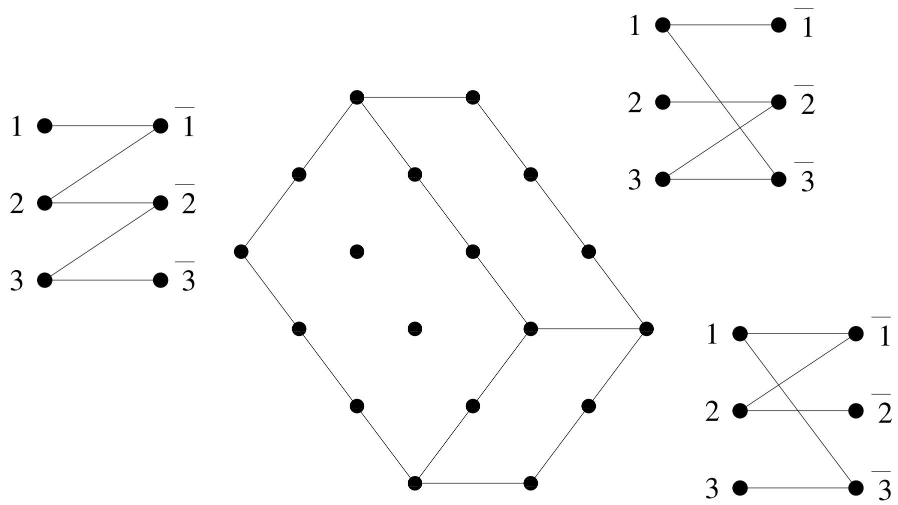

Figure 2: A fine mixed subdivision of $P_G(1, 2, 3)$.

We can say a bit more about the lattice points in each $\prod_T$:

Proposition 2.1 ([3], Proposition 14.12)**.** *Any lattice point of a fine Minkowski cell $\prod_T$ in $P_G(y_1, \cdots, y_m)$ is uniquely expressed (within $\prod_T$) as $p_1 + \cdots + p_m$ where p_i is a lattice point in $y_i \Delta_{T_i}$.*

3. Semi-polytope decomposition

A mixed subdivision of P_G divides the polytope into cells. In this section, we show that from a mixed subdivision of P_G, one can obtain a way to decompose the set of lattice points of P_G^-.

Definition 3.1 ([3], Definition 11.2)**.** *The* **trimmed generalized permutohedron** P_G^- *is defined as:*

$$P_G^-(y_1, \dots, y_m) := \{x \in \mathbb{R}^n | x + \Delta_{[n]} \subseteq P_G\}.$$

This is a more general class of polytopes than generalized permutohedra $P_G(y_1, \dots, y_m)$. With a slight abuse of notation, we will let $I \setminus j$ stand for $I \setminus \{j\}$.

Definition 3.2 ([3], Theorem 11.3)**.** *The coordinate* **semi-simplices** *are defined as*

$$\Delta^*_{I,j} = \Delta_I \setminus \Delta_{I \setminus j}$$

for $j \in I \subseteq [n]$.

For each cell $\prod_T$, we are going to turn it into a ***semi-polytope*** of the form $y_1\Delta^*_{J_1,j_1} + \cdots + y_m\Delta^*_{J_m,j_m}$. This will involve deciding which cell takes ownership of the lattice points on several cells at the same time.

We denote the point $((m-1)c + \sum_i y_i, -c, \ldots, -c)$ for c sufficiently large as ∞_1. For a facet of a polytope, we say that it is ***negative*** if the defining hyperplane of the facet (inside the space $\sum_{i\in[n]} x_i = \sum_{j\in[m]} y_j$ which the polytope lies in) separates the point ∞_1 and the interior of the polytope. Otherwise, we say that it is ***positive***. We will say that a point of a polytope is ***good*** if it is not on any of the positive facets of the polytope.

Lemma 3.1. *Fix T, a spanning tree of $G \subseteq K_{m,n}$. Let T_i be the set of neighbors of left vertex i. There exists a unique element t_i in T_i such that there exists a path to $\bar{1}$ not passing through i.*

Proof. There exists such an element since T is a spanning tree of T. There cannot be more than one such element since otherwise, we get a cycle in T. □

In particular when $\bar{1} \in T_i$, we have $t_i = \bar{1}$.

Lemma 3.2. *Let $\prod_T$ be a fine mixed cell. Removing the positive facets gives us $\sum_i \Delta^*_{T_i,t_i}$.*

To prove this, we first introduce a tool that will be useful for identifying which hyperplanes the facets lie on. Let $\prod_T$ be a fine mixed cell so T is a spanning tree. For any edge e of T that is not connected to a leaf on the left side, $T \setminus e$ has two components. Let I_e denote the set of right vertices of a component that contains $\bar{1}$. Let c_e be the sum of y_i's for left vertices contained in that component. Notice that I_e cannot be $[n]$ since otherwise e would have a leaf as its left endpoint.

Lemma 3.3. *Let $\prod_T$ be a fine mixed cell. For any edge e of T that is not connected to a leaf on the left side, $\prod_{T\setminus e}$ is a facet of $\prod_T$ that lies on $\sum_{j\in I_e} x_j = c_e$. If the right endpoint of e is in I_e, then $\prod_T$ lies in half-space $\sum_{j\in I_e} x_j \geq c_e$. Otherwise it lies in $\sum_{j\in I_e} x_j \leq c_e$*

Proof. The dimension difference between $\prod_T$ and $\prod_{T\setminus e}$ is at most one, and all endpoints of $\prod_{T\setminus e}$ lie on $\sum_{j\in I_e} x_j = c_e$. If the right endpoint of e is in I_e, that means we can find a point $x \in \prod_T$ using e so that $\sum_{j\in I_e} x_j > c_e$. If not, that means we can find a point $x \in \prod_T$ using e so that $\sum_{j\in I_e} x_j < c_e$. □

Proof of Lemma 3.2. If $\prod_{T\setminus e}$ is a positive facet of $\prod_T$, then Lemma 3.3 tells us that the right endpoint of $e = (i, \bar{j})$ is in I_e. From definition of t_i, we have $\bar{j} = t_i$. In other words we are removing sets of form $\Delta_{T_1} + \cdots + \Delta_{T_i\setminus t_i} + \cdots + \Delta_{T_m}$. At the end we end up with $\sum_i(\Delta_{T_i} \setminus \Delta_{T_i\setminus t_i})$.

□

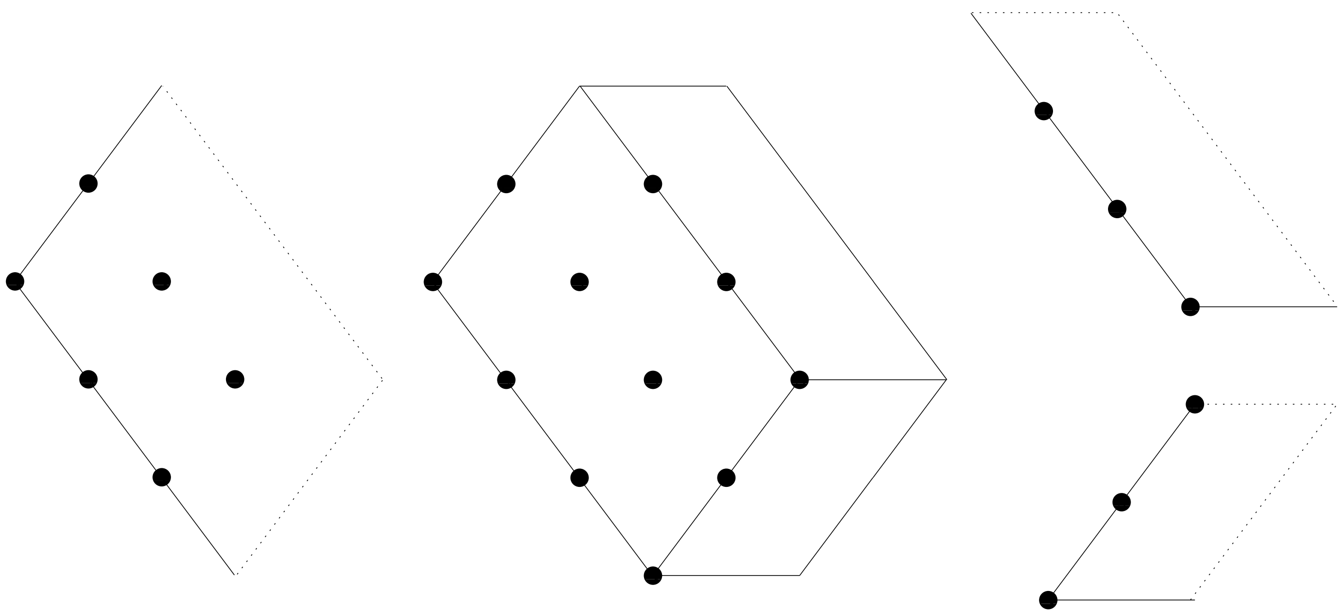

Figure 3: A semi-polytope decomposition of $P^-_G(1, 2, 3)$.

Let $\prod_T$ and $\prod'_T$ be two cells inside a mixed subdivision of P_G that share a facet F. The sign of F in $\prod_T$ and the sign of F in $\prod'_T$ has to be different, since ∞_1 can be on exactly one side of the defining hyperplane of F. This implies that all lattice points of P_G are good in at most one cell of P_G.

Lemma 3.4. *$p \in P_G^- \cap \mathbb{Z}^n$ if and only if $p + e_1$ is a good point of P_G.*

Proof. Having $p \in P_G^- \cap \mathbb{Z}^n$ implies from definition that $p + e_i \in P_G$ for each $i \in [n]$. Assume for sake of contradiction $p + e_1$ is on some positive facet $\sum_{i \in I} x_i = c_I$ with $1 \in I$. Any $p + e_i$ is either on that facet or is on the same side as ∞_1. Hence $I = [n]$ and we get a contradiction.

Now look at the case when $p + e_1$ is a good point of P_G. Assume for the sake of contradiction that $p + e_j$ is not in P_G for some $j \in [n]$. Then $p + e_1$ is on a facet of P_G, whose corresponding hyperplane is given by $\sum_{i \in I} x_i = c_I$ where $1 \in I$ and $j \notin I$. This hyperplane separates the interior of P_G with $p + e_j$. Since $j \notin I$, the point ∞_1 has to be on opposite side of $p + e_j$. This is a positive facet and we get a contradiction.

□

Combining what we have so far, we state the main result of our note on how to do the semi-polytope decomposition with an explicit way to obtain each of the semi-polytopes:

Theorem 3.1. *Identify the lattice points of P_G^- with lattice points not lying on any of the positive facets of P_G via the map $p \to p + e_1$ (as in Lemma 3.4). Pick any full mixed subdivision of P_G. For each cell $\prod_T$, construct a semi-polytope by $\sum_i \Delta^*_{T_i, t_i}$ (where t_i is chosen as in Lemma 3.1). Then the (disjoint) union of the semi-polytopes is exactly P_G minus the positive facets. Each lattice point of P_G^- with the above identification is contained in exactly one semi-polytope.*

An example of a semi-polytope decomposition of the generalized permutohedron considered in Figure 1 and in Figure 2 is given in Figure 3.

4. Application to Erhart theory

In this section we show how the semi-polytope decomposition can be used in Erhart theory as guided in [3]. Given any subgraph T in G, define the ***left degree vector*** $ld(T) = (d_1 - 1, \cdots, d_n - 1)$ and the ***right degree vector*** $rd(T) = (d'_1 - 1, \cdots, d'_m - 1)$ where d_i and d'_j are the degree of the vertex i and $\bar{j}$ respectively. The raising powers are defined as $(y)_a := y(y+1)\cdots(y+a-1)$ for $a \geq 1$ and $(y)_0 := 1$.

Corollary 4.1. *Fix a fine mixed subdivision of $P_G(y_1, \ldots, y_m)$ where y_i's are nonnegative integers. The number of lattice points in the trimmed generalized permutohedron $P_G^-(y_1, \ldots, y_m)$ equals*

$$\sum_{(a_1, \ldots, a_m)} \prod_i \frac{(y_i)_{a_i}}{a_i!}$$

where the sum is over all left-degree vectors of fine mixed cells inside the subdivision.

Proof. Obtain a semi-polytope decomposition as in Theorem 3.1. Then each lattice point of $P_G^-(y_1, \ldots, y_m)$ is in exactly one semi-polytope. The claim follows since the number of lattice points of a semi-polytope $y_1 \Delta^*_{T_1, t_1} + \cdots + y_m \Delta^*_{T_m, t_m}$ (thanks to Proposition 2.1, different sum gives a different point) is given by

$$\prod_i \frac{(y_i)_{|T_i|-1}}{(|T_i| - 1)!}.$$

□

The expression in Corollary 4.1 is called the ***Generalized Erhart polynomial*** of $P_G^-(y_1, \ldots, y_m)$ by [3]. As foretold in [3], Theorem 3.1 gives us a pure counting proof of Theorem 11.3 of [3].

Definition 4.1 ([3], Definition 9.2)**.** *Let us say that a sequence of nonnegative integers $(a_1, \cdots, a_m)$ is a G-**draconian sequence** if $\sum a_i = n - 1$ and, for any subset $\{i_1 < \cdots < i_k\} \subseteq [m]$, we have $|I_{i_1} \cup \cdots \cup I_{i_k}| \geq a_{i_1} + \cdots + a_{i_k} + 1$.*

Theorem 4.1 ([3], Theorem 11.3)**.** *For nonnegative integers $y_1, \ldots, y_m$, the number of lattice points in the trimmed generalized permutohedron $P_G^-(y_1, \ldots, y_m)$ equals*

$$\sum_{(a_1, \ldots, a_m)} \prod_i \frac{(y_i)_{a_i}}{a_i!},$$

where the sum is over all G-draconian sequences $(a_1, \ldots, a_m)$.

Proof. Thanks to Corollary 4.1, all we need to do is show that the set of G-draconian sequences is exactly the set of left-degree vectors of the cells inside a fine mixed subdivision of P_G. Lemma 14.9 of [3] tells us that the right degree vectors of the fine cells is exactly the set of lattice points of $P^-_{G^*}(1,\dots,1)$ where G^* is obtained from G by switching left and right vertices. Then Lemma 11.7 of [3] tells us that the set of G-draconian sequences is exactly the set of lattice points of $P^-_{G^*}(1,\dots,1)$. $\square$

This approach has an advantage that the generalized Erhart polynomial is obtained from a direct counting method, without using any comparison of formulas.

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