Recent lower bounds for geometric-arithmetic index

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Abstract

Although the concept of geometric-arithmetic indices has been introduced in the chemical graph theory recently, these indices have already been proved to be useful. After the excellent survey [K. C. Das, I. Gutman, B. Furtula, Survey on geometric-arithmetic indices of graphs, *MATCH Commun. Math. Comput. Chem.* **65** (2011) 595–644] on these indices, lots of papers have been (and are being) published on the (first) geometric-arithmetic index. The present survey tries to collect the lower bounds of the geometric-arithmetic index appeared after the publication of the mentioned survey.

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1. Introduction

A single number which represents a chemical structure in graph-theoretical terms via the molecular graph is called a *molecular descriptor*; besides, if it correlates with a molecular property, it is called *topological index* and it is used to understand physicochemical properties of chemical compounds. The interest of topological indices lies in the fact that they synthesize some of the properties of a molecule into a single number. With this in mind, hundreds of topological indices have been introduced and studied so far; it is worth noting the seminal work by Wiener in which he used the sum of all shortest-path distances of a (molecular) graph in order to model physical properties of alkanes (see [75]).

Topological indices based on end-vertex degrees of edges play a vital role in mathematical chemistry and some of them are recognized tools in chemical research. Probably, the very first such descriptor is the Platt index [56] and the best known among such descriptors are the Randić connectivity index and the Zagreb indices.

The first and second Zagreb indices, denoted by M_1 and M_2 , were appeared in [34] and [32] respectively:

$$M_1(G) = \sum_{u \in V(G)} d_u^2, \qquad M_2(G) = \sum_{uv \in E(G)} d_u d_v,$$

where uv denotes the edge of the graph G connecting the vertices u and v, and d_u is the degree of the vertex u.

There is a vast amount of research on the Zagreb indices. For details of their chemical applications and mathematical theory see [27], [28], [30], and the references therein.

The Randić connectivity index is defined in [58] as

$$R(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d_u d_v}}.$$

Over one thousand papers and a couple of books dealing with this molecular descriptor have been written (for example, see [29, 37, 38] and the references cited therein).

In [6], [40], [39], [48], the first and second variable Zagreb indices are defined as

$$M_1^{\alpha}(G) = \sum_{u \in V(G)} d_u^{\alpha}, \qquad M_2^{\alpha}(G) = \sum_{uv \in E(G)} (d_u d_v)^{\alpha},$$

with $\alpha \in \mathbb{R}$.

The concept of *variable molecular descriptors* was proposed as a new way of characterizing heteroatoms in molecules (see [57], [59]), but also to assess the structural differences (e.g., the relative role of carbon atoms of acyclic and cyclic

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parts in alkylcycloalkanes [60]). The idea behind the variable molecular descriptors is that the variables are determined during the regression; this allows to make the standard error of the estimate for a particular property (targeted in the study) as small as possible (see, e.g., [48]).

Gutman and Tošović [33] tested the correlation abilities of 20 vertex-degree-based topological indices used in the chemical literature for the case of standard heats of formation and normal boiling points of octane isomers. It is noteworthy that the second variable Zagreb index M_2^{α} with exponent $\alpha = -1$ (and to a lesser extent with exponent $\alpha = -2$) performs significantly better than the Randić index ($R = M_2^{-1/2}$).

The second variable Zagreb index is used in the structure-boiling point modeling of benzenoid hydrocarbons [52]. Also, variable Zagreb indices exhibit a potential applicability for deriving multi-linear regression models [14]. Various properties and relations of these indices are discussed in several papers.

Note that M_1^2 is the first Zagreb index M_1 , M_1^{-1} is the inverse index ID, M_1^3 is the forgotten index F, etc.; also, $M_2^{-1/2}$ is the usual Randić index, M_2^1 is the second Zagreb index M_2 , M_2^{-1} is the modified Zagreb index, etc.

The general sum-connectivity index was defined by Zhou and Trinajstić in [79] as

$$\chi_{\alpha}(G) = \sum_{uv \in E(G)} (d_u + d_v)^{\alpha}.$$

Note that χ_1 is the first Zagreb index M_1 , $2\chi_{-1}$ is the harmonic index H, $\chi_{-1/2}$ is the sum-connectivity index χ , etc.

A class of molecular descriptors, named as "geometric-arithmetic indices" ($GA_{general}$) are defined (see [20] or [73]) as

$$GA_{general}(G) = \sum_{uv \in E(G)} \frac{\sqrt{Q_u Q_v}}{\frac{1}{2}(Q_u + Q_v)},$$

where Q_u is some quantity associated with the vertex $u \in V(G)$. The name of this class of indices is evident from their definition. Namely, indices belonging to this group are calculated as the ratio of geometric and arithmetic means of some properties of adjacent vertices u and v.

The first member of this class is the so-called (first) geometric-arithmetic index defined in [73] as

$$GA_1(G) = GA(G) = \sum_{uv \in E(G)} \frac{\sqrt{d_u d_v}}{\frac{1}{2}(d_u + d_v)}$$

Although GA was introduced barely a decade ago, there are already many papers dealing with this index (see, e.g., [9], [11], [10], [73] and the references therein). The results in [11, p.598] show that the GA index gathers the same information on the molecule under study as other geometric-arithmetic indices. So, we will focus in this survey on GA.

The number of possible benzenoid hydrocarbons is huge, although only about 1,000 of them have been identified so far. As an example, there are as many as $5.85 \cdot 10^{21}$ benzenoid hydrocarbons with exactly 35 benzene rings [71]. Therefore, the ability to model their physicochemical properties can be most helpful in order to foresee characteristics of currently unknown species. The predicting ability of the *GA* index compared with Randić index is reasonably better (see [11, Table 1]). The graphic in [11, Fig.7] (from [11, Table 2], [68]) shows that there exists a good linear correlation between *GA* and the heat of formation of benzenoid hydrocarbons (the correlation coefficient is equal to 0.972).

Furthermore, in the case of standard enthalpy of vaporization, GA index improves the performance of Randić index by more than 9%. That is why one can think that GA index should be considered in the research on QSPR/QSAR.

A main topic in the study of topological indices is to find bounds of the indices. After the excellent survey in 2011 on geometric-arithmetic indices (see [11]), lots of papers have been (and are being) published on the (first) geometric-arithmetic index. This survey try to collect most lower bounds of the GA index since 2011. Furthermore, we have improved slightly some of them by removing some hypotheses, such as lower bounds on the minimum degree or the connectedness. Also, we include some new inequalities.

Throughout this survey, G = (V(G), E(G)) denotes a (non-oriented) finite simple (without multiple edges and loops) graph without isolated vertices (i.e., the minimum degree of G is at least 1). We denote by Δ, δ, n, m the maximum degree, the minimum degree and the cardinality of the set of vertices and edges of G, respectively. Thus, $\Delta \ge \delta \ge 1$, $n \ge 2$ and $m \ge 1$.

Recall that a (Δ, δ) -biregular graph (or simply a biregular graph) is a bipartite graph for which any vertex in one side of the given bipartition has degree Δ and any vertex in the other side of the bipartition has degree δ .

Along this work, if two graphs G_1 and G_2 are isomorphic, we will write $G_1 \cong G_2$.

The outline of this paper is as follows: Section 2 collects some lower bounds of the geometric-arithmetic index GA(G) in terms of some parameters of the graph G (such as the total number of vertices or edges of G, the minimum

or maximum degree, the number of pendant or fully-connected vertices, the chromatic number of G or its hyperbolicity constant). Section 3 compiles some lower bounds of GA(G) involving some matrices (such as the adjacency matrix, the GA-adjacency matrix, the GA-Laplace matrix or the sum-connectivity matrix) and their respective eigenvalues or traces. Section 4 relates some lower bounds of GA(G) with other topological indices; among them, Zagreb, Randić, harmonic, sum-connectivity, atom-bond connectivity or forgotten indices can be found, as well as some combinations of them. In Section 5 some lower bounds of GA(G) in terms of subgraphs of G can be found. Finally, Section 6 collects lower bounds for the line graph $\mathcal{L}(G)$ associated with G.

2. Lower bounds of GA(G) in terms of parameters of G

Probably the best known lower bound of GA(G) is

$$GA(G) \ge \frac{2m\sqrt{\Delta\delta}}{\Delta+\delta},$$
 (1)

where G is a graph with m edges, maximum degree Δ and minimum degree δ [9]. This inequality is interesting by itself and, also, it is used in order to obtain other inequalities.

The following inequalities can be found in [73, Theorem 2], [50, Proposition 2], [11, pp. 609–610],

$$GA(G) \ge \frac{2(n-1)^{3/2}}{n}, \qquad GA(G) \ge \frac{2m}{n},$$
 (2)

where G is a graph with n vertices and m edges. The first inequality requires the graph to be connected.

The next result provides a lower bound of GA(G) depending just on the number of vertices and edges. Besides, it improves both inequalities in (2).

Theorem 2.1. [63, Theorem 2.4] Let G be a graph with n vertices and m edges. Then

$$GA(G) \ge \frac{2m\sqrt{n-1}}{n}$$
,

and the equality is attained if and only if G is a star graph.

Theorem 2.4 of [63] was proved for connected graphs, but the argument in its proof also works if G is not connected.

The inequality in Theorem 2.1 is a consequence of (1).

Next, we include a lower bound of GA(G) depending just on the number of vertices.

Proposition 2.1. [55, Proposition 2] If G is a graph with n vertices, m edges and maximum degree Δ such that $\Delta \leq n-2$, then

$$GA(G) > \frac{2m\sqrt{n-2}}{n-1} \ge 2\sqrt{n-2}$$

The first inequality in Proposition 2.1 is a consequence of (1).

Proposition 2.2. [46, Proposition 5] If G is a graph with n vertices, m edges and minimum degree δ then

$$GA(G) \ge \frac{2m\sqrt{(n-1)\delta}}{n+\delta-1}$$
.

The equality is attained if and only if G is either a complete graph or a star graph.

The inequality in Proposition 2.2 is also a consequence of (1).

The following theorem provides a lower bound of GA(G) for every graph G with minimum degree $\delta \ge k$, for any fixed $k \ge 2$. This result improves the first inequality in (2).

Theorem 2.2. [63, Theorem 2.5] Consider any graph G with n vertices and minimum degree $\delta \ge k \ge 2$.

1. If $n \leq 10$, then

$$GA(G) \ge \frac{nk}{2}$$

2. If $n \ge 11$, then

$$GA(G) \ge \min\left\{\frac{nk}{2}, \frac{(k+1)\sqrt{k}(n-1)^{3/2}}{n-1+k}\right\}$$

Theorem 2.5 of [63] was proved for connected graphs, but the argument in its proof also works if G is not connected.

The following theorem provides lower bounds of GA(G) depending just on the minimum and maximum degree of G. Their proofs use (1) and the lower bounds on the number of edges in terms on the minimum and maximum degree of G in [44, Proposition 2.5].

Theorem 2.3. [44, Theorem 2.7] Let G be a graph with minimum degree δ and maximum degree Δ . Then

$$GA(G) \ge \Delta(\delta+1)\frac{\sqrt{\delta\Delta}}{\delta+\Delta}$$

with equality if and only if either $\delta = 1$ and G is a star graph or $\delta = \Delta$ and G is a complete graph. Furthermore, if $\Delta(\delta + 1)$ is odd, then

$$GA(G) \ge \left(\Delta(\delta+1)+1\right) \frac{\sqrt{\delta\Delta}}{\delta+\Delta}$$

Proposition 2.3. [54, Proposition 3] Let G be a graph with m edges, n vertices and maximum degree Δ . Then

$$GA(G) \ge \frac{2}{\Delta} \frac{m^2}{n}.$$

The equality is attained for any regular graph.

The bound in Proposition 2.3 is always better than the one in Proposition 4.4 since $\delta \leq n-1$. Also, it improves the second inequality in (2), since $m \geq \Delta$.

Next result is a consequence of Proposition 4.4.

Corollary 2.1. [54, Corollary 3] Let G be a graph with $n \ge 3$ vertices, m edges, minimum degree δ and maximum degree Δ . Then

$$GA(G) \ge \frac{2\delta m^2}{\Delta(\delta+1)(n-1)}.$$

Theorem 2.4. [13, Theorem 1] Let G be a connected graph with n vertices and minimum degree δ . If $\delta \geq \lceil \delta_0 \rceil$, where $\delta_0 = q_0(n-1)$ and $q_0 \approx 0.088$ is the unique positive root of the equation $q\sqrt{q} + q + 3\sqrt{q} - 1 = 0$, then

$$GA(G) \ge \frac{\delta n}{2}.$$

If δ or n are even, this value is attained by any regular graph.

The *chromatic number* of a graph G, denoted by $\mathcal{C}(G)$, is the smallest number of colors needed to color the vertices of G so that no two adjacent vertices share the same color.

Corollary 2.2. [3, Corollary 2.11] Let G be a connected graph with minimum degree $\delta \geq 2$. Then

$$GA(G) \ge \frac{\delta}{2} \mathcal{C}(G),$$

with equality if and only if G is a complete graph.

Motivated by the above result, some experiments with the help of AutoGraphiX were conducted in [4] that led the authors to state an analogous result replacing δ by the average degree $\overline{\delta}$ (see [4, Conjecture 3.2]). Some other conjectures about lower bounds for the geometric-arithmetic index stated in the same article were later disproved in [7].

Given positive integers $\delta \leq \Delta$, define $\mathcal{G}_{\delta,\Delta}$ as the set of graphs G with $\Delta + 1$ vertices, minimum degree δ , maximum degree Δ and such that:

- 1. *G* is isomorphic to the complete graph with $\Delta + 1$ vertices, if $\delta = \Delta$,
- 2. there are Δ vertices with degree δ , if $\delta < \Delta$ and $\Delta(\delta + 1)$ is even,
- 3. there are $\Delta 1$ vertices with degree δ and a vertex with degree $\delta + 1$, if $\delta < \Delta 1$ and $\Delta(\delta + 1)$ is odd,

4. there are $\Delta - 1$ vertices with degree δ and two vertices with degree Δ , if $\delta = \Delta - 1$ and Δ is odd (and thus $\Delta(\delta+1)$ is odd).

Remark 2.1. Since every graph $G \in \mathcal{G}_{\delta,\Delta}$ has maximum degree Δ and $|V(G)| = \Delta + 1$, it follows that every $G \in \mathcal{G}_{\delta,\Delta}$ is connected.

Theorem 2.5. [44, Theorem 2.12] Let G be a graph with minimum degree δ and maximum degree $\Delta \geq 2$. If

$$\frac{2\sqrt{\delta\Delta}}{\delta+\Delta} \ge \frac{\Delta(\delta-1)}{\Delta(\delta-1)+2},\tag{3}$$

then

$$GA(G) \ge \frac{2\Delta\sqrt{\delta\Delta}}{\delta+\Delta} + \frac{\Delta(\delta-1)}{2}.$$
 (4)

Furthermore, if $\Delta(\delta + 1)$ *is odd,*

$$\frac{2\sqrt{\delta\Delta}}{\delta+\Delta} \ge \frac{\Delta(\delta-1)}{\Delta(\delta-1)+2} \quad and \quad \frac{3\sqrt{\delta\Delta}}{\delta+\Delta} + \delta - \frac{1}{2} \ge \frac{2\sqrt{(\delta+1)\Delta}}{\delta+1+\Delta} + \frac{2\delta\sqrt{\delta(\delta+1)}}{2\delta+1}, \tag{5}$$

then

$$GA(G) \ge \frac{2(\Delta-1)\sqrt{\delta\Delta}}{\delta+\Delta} + \frac{2\sqrt{(\delta+1)\Delta}}{\delta+1+\Delta} + \frac{2\delta\sqrt{\delta(\delta+1)}}{2\delta+1} + \frac{(\Delta-2)(\delta-1)-1}{2}.$$
(6)

If Δ and δ satisfy (3), then the equality in (4) is attained if and only if $\Delta(\delta + 1)$ is even and $G \in \mathcal{G}_{\delta,\Delta}$. If Δ and δ satisfy (5) and $\Delta(\delta + 1)$ is odd, then the equality in (6) is attained if and only if $G \in \mathcal{G}_{\delta,\Delta}$.

A connected graph with maximum degree at most four is a *chemical graph*, and it is usually employed to represent hydrocarbons [70]. Theorem 2.5 allows to obtain sharp inequalities for chemical graphs.

Corollary 2.3. [44, Corollary 2.14] Let G be a chemical graph with minimum degree δ and maximum degree Δ . If $(\delta, \Delta) \neq (2, 3)$, then

$$GA(G) \ge \frac{2\Delta\sqrt{\delta\Delta}}{\delta+\Delta} + \frac{\Delta(\delta-1)}{2},$$

with equality if and only if $G \in \mathcal{G}_{\delta,\Delta}$. If $(\delta, \Delta) = (2, 3)$, then

$$GA(G) \geq \frac{2(\Delta-1)\sqrt{\delta\Delta}}{\delta+\Delta} + \frac{2\sqrt{(\delta+1)\Delta}}{\delta+1+\Delta} + \frac{2\delta\sqrt{\delta(\delta+1)}}{2\delta+1} + \frac{(\Delta-2)(\delta-1)-1}{2} = \frac{8\sqrt{6}}{5} + 1,$$

with equality if and only if $G \in \mathcal{G}_{2,3}$.

Corollary 2.4. [44, Corollary 2.15] *Let G* be a graph with minimum degree δ and maximum degree $\Delta = \delta + h \ge 2$. If $(16 - h^2)\Delta^3 + (2h^3 + 2h^2 - 32h - 16)\Delta^2 + (-h^4 - 2h^3 + 15h^2 + 16h + 16)\Delta - 16h \ge 0$, then

$$GA(G) \ge \frac{2\Delta\sqrt{\Delta(\Delta-h)}}{2\Delta-h} + \frac{\Delta(\Delta-h-1)}{2}.$$

Corollary 2.5. [44, Corollary 2.16] Let G be a graph with minimum degree δ and maximum degree $\Delta = \delta + h \ge 2$. If we have

- 1. h = 0 or h = 1, for every $\Delta \ge 2$,
- 2. h = 2, for every $\Delta \geq 3$,
- 3. h = 3, for every $\Delta \ge 4$,
- 4. h = 4, for every $\Delta \ge 5$,
- 5. h = 5, for every $\Delta \in \{6, 7, 8\}$,
- 6. h = 6, for every $\Delta \in \{7, 8\}$,
- 7. $h \geq 7$ and $\Delta = h + 1$,

then

$$GA(G) \ge \frac{2\Delta\sqrt{\Delta(\Delta-h)}}{2\Delta-h} + \frac{\Delta(\Delta-h-1)}{2}.$$

Corollary 2.6. [44, Corollary 2.17] Let G be a graph with maximum degree $\Delta \ge 2$ and minimum degree $\delta = \Delta - 1$. Then

$$\begin{split} & GA(G) \geq \frac{2\Delta\sqrt{\Delta(\Delta-1)}}{2\Delta-1} + \frac{\Delta(\Delta-2)}{2}, \quad \text{if } \Delta \text{ is even,} \\ & GA(G) \geq \frac{4(\Delta-1)\sqrt{\Delta(\Delta-1)}}{2\Delta-1} + \frac{(\Delta-2)^2-1}{2} + 1, \quad \text{if } \Delta \text{ is odd,} \end{split}$$

with equalities if and only if $G \in \mathcal{G}_{\Delta-1,\Delta}$.

Corollary 2.7. [44, Corollary 2.18] Let G be a graph with minimum degree δ and maximum degree $2 \le \Delta \le 8$. Then

$$GA(G) \ge \frac{2\Delta\sqrt{\delta\Delta}}{\delta+\Delta} + \frac{\Delta(\delta-1)}{2}.$$

Let us denote by $K_{\delta,\Delta}$ the complete bipartite graph with a partition K_1 , K_2 with δ and Δ vertices respectively. Notice that the vertices in K_1 have degree Δ and the vertices in K_2 have degree δ . It has been proved in [65] that $GA(K_{\delta,\Delta}) = \frac{2\delta\Delta\sqrt{\delta\Delta}}{\delta+\Delta}$.

Theorem 2.6. [44, Theorem 2.21] Let G be a graph with minimum degree 2 and maximum degree $\Delta \geq 28$. Then

$$GA(G) \geq \frac{4\Delta\sqrt{2\Delta}}{\Delta+2},$$

and the equality is attained if and only if $G \cong K_{2,\Delta}$.

Theorem 2.6 shows that inequalities (3) and (5) in Theorem 2.5 do not hold for every graph.

Given any odd integer $\Delta \geq 3$, let us define H_{Δ} as the graph with minimum degree 2, maximum degree Δ , $|V(H_{\Delta})| = \Delta + 1$, and such that there are 2 vertices, x_0, x_1 with degree Δ which are adjacent and $\Delta - 1$ vertices with degree 2: x_2, \ldots, x_{Δ} . Note that

$$GA_1(H_\Delta) = 2(\Delta - 1)\frac{2\sqrt{2\Delta}}{2+\Delta} + 1.$$
(7)

The next result shows that the conclusion of Theorem 2.6 does not hold for $\Delta < 28$.

Proposition 2.4. [44, Proposition 2.22] For any integer $2 \le \Delta \le 27$, if $G \in \mathcal{G}_{2,\Delta}$, then

$$GA(G) < GA(K_{2,\Delta}), \quad \text{if } \Delta \text{ is even,}$$

 $GA(H_{\Delta}) < GA(K_{2,\Delta}), \quad \text{if } \Delta \text{ is odd.}$

Let us consider an ordering of the vertices in G where u < v implies that $d_u \leq d_v$. Let us assume an orientation of the edges where uv is always taken with the orientation given by the ordering u < v. Let $k = \Delta - \delta$, let m_i be the number of oriented edges whose tail is a vertex with degree $\delta + i$ and m'_i the number of oriented edges whose head is a vertex with degree $\delta + i$ for $0 \leq i \leq k$. Moreover, let a_i be the number of edges whose tail is a vertex with degree $\delta + i$ and whose head is a vertex with degree at least $\delta + i + 1$ with $0 \leq i \leq k - 1$, let b_i the number of edges whose head is a vertex with degree $\delta + i$ and whose tail is a vertex with degree at most $\delta + i - 1$ with $1 \leq i \leq k$, and c_i the number of edges joining two vertices with degree $\delta + i$ with $0 \leq i \leq k$. Notice that $m_i = a_i + c_i$ and $m'_i = b_i + c_i$ for every $0 \leq i \leq k$, $m_k = c_k$ and $m'_0 = c_0$.

Define the classes of graphs \mathcal{G}_1 and \mathcal{G}_2 as follows. \mathcal{G}_1 is the set of graphs G such that if $uv \in E(G)$, then $d_u = d_v$ or $\max\{d_u, d_v\} = \Delta$, where Δ is the maximum degree of G. \mathcal{G}_2 is the set of graphs G such that if $uv \in E(G)$, then $d_u = d_v$ or $\min\{d_u, d_v\} = \delta$, where δ is the minimum degree of G.

Proposition 2.5. [46, Proposition 2] Let G be a graph with minimum degree δ and maximum degree $\Delta > \delta$. Then

$$GA(G) \ge \sum_{i=0}^{k} c_i + \sum_{i=0}^{k-1} \frac{2a_i \sqrt{\Delta(\delta+i)}}{\Delta + \delta + i},$$
(8)

and

$$GA(G) \ge \sum_{i=0}^{k} c_i + \sum_{i=1}^{k} \frac{2b_i \sqrt{\delta(\delta+i)}}{2\delta+i}.$$
(9)

The equality in (8) is attained if and only if $G \in \mathcal{G}_1$. The equality in (9) is attained if and only if $G \in \mathcal{G}_2$.

Define the classes of graphs \mathcal{G}_1^0 and \mathcal{G}_2^0 as follows. \mathcal{G}_1^0 is the set of graphs G such that if $uv \in E(G)$, then $\max\{d_u, d_v\} = \Delta$, where Δ is the maximum degree of G. \mathcal{G}_2^0 is the set of graphs G such that if $uv \in E(G)$, then $\min\{d_u, d_v\} = \delta$, where δ is the minimum degree of G. It is clear that $\mathcal{G}_1^0 \subset \mathcal{G}_1$ and $\mathcal{G}_2^0 \subset \mathcal{G}_2$.

Corollary 2.8. [46, Corollary 2] Let G be a graph with minimum degree $\delta \ge 2$ and maximum degree $\Delta > \delta$. Then

$$GA(G) \ge \sum_{i=0}^{k} \frac{2m_i \sqrt{\Delta(\delta+i)}}{\Delta+\delta+i} = \sum_{i=0}^{k-1} \frac{2m_i \sqrt{\Delta(\delta+i)}}{\Delta+\delta+i} + m_k$$

and

$$GA(G) \ge \sum_{i=0}^k \frac{2m'_i \sqrt{\delta(\delta+i)}}{2\delta+i} = m'_0 + \sum_{i=1}^k \frac{2m'_i \sqrt{\delta(\delta+i)}}{2\delta+i}.$$

The first (respectively, second) equality is attained if and only if $G \in \mathcal{G}_1^0$ (respectively, $G \in \mathcal{G}_2^0$).

Since in a connected graph with at least 3 vertices there are no edges joining two vertices with degree 1, the following consequence holds.

Corollary 2.9. [46, Corollary 3] Let G be a connected graph with at least 3 vertices, minimum degree 1 and maximum degree Δ . Then

$$GA(G) \ge \sum_{i=0}^{k} \frac{2m_i \sqrt{\Delta(i+1)}}{\Delta + i + 1}$$

Given any graph G and $uv \in E(G)$, let us define the gradient of the edge uv as $\nabla_{uv} := |d_u - d_v|$.

Proposition 2.6. [46, Proposition 3] Let G be a graph with m edges and minimum degree δ . If $D = \max_{uv \in E(G)} \nabla_{uv}$, then

$$GA(G) \ge \frac{2m\sqrt{\delta(\delta+D)}}{2\delta+D}.$$
(10)

The equality is attained if and only if G is either regular or biregular.

Let E_0, \ldots, E_k (with $k = \Delta - \delta$) be a partition of the edges of G given by the gradient where $e \in E_i$ if $\nabla_e = i$ for each $0 \le i \le k$. Let e_i be the number of edges in E_i .

Proposition 2.7. [46, Proposition 4] Let G be a graph with minimum degree δ and maximum degree Δ . Then

$$GA(G) \ge \sum_{i=0}^{k} \frac{2e_i \sqrt{\delta(\delta+i)}}{2\delta+i}.$$
(11)

The equality is attained if and only if $G \in \mathcal{G}_2^0$.

In [45], the authors define the following class of graphs:

Given integers $0 < i \le \delta < \Delta$, let us define $\mathcal{H}^i_{\delta,\Delta}$ as the set of graphs H with minimum degree δ , maximum degree Δ , $|V(H)| = \Delta + 1$, and such that:

- 1. there are *i* vertices with degree Δ and $\Delta i + 1$ vertices with degree δ , if $(\Delta i + 1)(\delta i)$ is even,
- 2. there are *i* vertices with degree Δ , Δi vertices with degree δ and an additional vertex with degree $\delta + 1$ (possibly Δ if $\delta = \Delta 1$), if $(\Delta i + 1)(\delta i)$ is odd.

In that same paper, it is proved (see [45, Proposition 11]) that $\mathcal{H}^i_{\delta,\Delta} \neq \emptyset$.

Definition 2.1. A graph G with minimum degree δ and maximum degree Δ is minimal for GA if $GA(G) \leq GA(\Gamma)$ for every graph Γ with minimum degree δ and maximum degree Δ .

The following conjecture is presented in that same paper.

Conjecture 2.1. [45, Conjecture 13] Given any integers $1 < \delta < \Delta$, a graph G is minimal for GA_1 if and only if $G \in \mathcal{H}^i_{\delta,\Delta}$ for some $1 \le i \le \delta$.

Recall that a *pendant vertex* is a vertex with degree 1 and a *pendant edge* is an edge with a pendant vertex. Also, if the vertex set V is the disjoint union of two nonempty sets V_1 and V_2 , such that every vertex in V_1 has degree r and every vertex in V_2 has degree $s \neq r$, then G is said to be (r, s)-semiregular. When r = s then G is a regular graph of degree r.

Proposition 2.8. [5, Proposition 2] Let G be a graph with n vertices, m edges, h pendant vertices and such that the degree sequence of the non-pendant vertices satisfies $d_1 \ge d_2 \ge \ldots \ge d_{n-h}$. Let Δ_1 be the largest degree among vertices in pendant edges and Δ_2 the largest degree among vertices in non-pendant edges. Then

$$GA(G) \ge \frac{2}{\sqrt{\Delta_1}} \frac{hd_{n-h}}{1+d_{n-h}} + \frac{2}{\Delta_2} \frac{(m-h)^2}{n-h-\frac{h}{\Delta_1}}$$

The equality is attained by the star graph, all $(1, \Delta)$ -semiregular graphs and all regular graphs.

This gives the following:

Corollary 2.10. [5, Corollary 1] Let G be a graph with n vertices, m edges, h pendant vertices and such that the degree sequence of the non-pendant vertices satisfies $d_1 \ge d_2 \ge \ldots \ge d_{n-h}$. Let Δ_1 be the largest degree among vertices in pendant edges and Δ_2 the largest degree among vertices in non-pendant edges. Then

$$GA(G) \ge \frac{2}{\sqrt{\Delta_1}} \frac{hd_{n-h}}{1+d_{n-h}} + \frac{(m-h)d_{n-h}}{\Delta_2}.$$

The equality is attained by the star graph, all $(1, \Delta)$ -semiregular graphs and all regular graphs.

Remark 2.2. In [5] the authors prove that the bound in Corollary 2.10 is better than the bound in Proposition 2.8 if and only if

$$d_{n-h} > \frac{2(m-h)}{n-h-\frac{h}{\Delta_1}}$$

Also, it is shown that this condition can be satisfied.

Recall that if G is a graph with n vertices, a *fully connected* vertex is a vertex with degree n - 1.

Proposition 2.9. [5, Proposition 3] Let G be a graph with n vertices and h > 1 fully connected vertices. Denote by Δ_3 the largest degree among the not-fully connected vertices and by m' the cardinal of the set of edges joining not-fully connected vertices. Then

$$GA(G) \ge \frac{h(h-1)}{2} + \frac{2(n-h)\sqrt{(n-1)h^3}}{n-1+\Delta_3} + \frac{hm'}{\Delta_3}$$

The equality is attained by the complete graph K_n and all (n-1,h)-semiregular graphs.

The authors make the observation that the bound in Proposition 2.9 performs better than the bound in Proposition 2.3 among graphs with h > 1 fully connected vertices.

The study of Gromov hyperbolic graphs is a subject of increasing interest, both in pure and applied mathematics (see e.g. [47] and the references cited therein). We say that a graph is *t*-hyperbolic $(t \ge 0)$ if any side of every geodesic triangle is contained in the *t*-neighborhood of the union of the other two sides. We define the hyperbolicity constant $\delta(G)$ of a graph G as the infimum of the constants $t \ge 0$ such that G is *t*-hyperbolic. For this purpose, every edge is taken of length 1.

The following inequality relates the geometric-arithmetic index with the hyperbolicity constant $\delta(G)$.

Theorem 2.7. [63, Theorem 2.7] For any connected graph G that is not a tree

$$GA(G) \ge \frac{2(4\delta(G) - 1)^{3/2}}{4\delta(G)}$$

The proof of Theorem 2.7 uses the first inequality in (2) and some estimates of the hyperbolicity constant.

For the next result, some notation will be needed. Recall that C_n is the cycle with n vertices. I_n will denote the graph consisting of n individual vertices and no edges and E_{2n} will represent the graph consisting of 2n vertices and n edges in which no two edges are adjacent. The graph $G \bigvee H$ is obtained from the disjoint graphs G and H by connecting each vertex of G with each vertex of H.

Theorem 2.8. [67, Theorem 1] Let G be a connected graph with n vertices and minimum degree 2. Then,

$$GA(G) \ge \begin{cases} n , & \text{if } n \le 24, \\ 24.79 , & \text{if } n = 25, \\ g(n) , & \text{if } n \ge 26, \end{cases}$$

where

$$g(n) = \frac{4(n-2)\sqrt{2(n-2)}}{n}$$

These values are attained on the C_n for $n \le 24$, $I_1 \bigvee E_{24}$ for n = 25 and $I_2 \bigvee I_{n-2}$ (which is bipartite graph $K_{2,n-2}$) for n = 26.

3. Lower bounds of GA(G) in terms of matrices

Given a graph G, let us define the GA-adjacency matrix A with entries

$$a_{uv} := \begin{cases} \frac{2\sqrt{d_u d_v}}{d_u + d_v}, & \text{ if } uv \in E(G), \\ 0, & \text{ otherwise.} \end{cases}$$

Let us also define \mathcal{D} as the diagonal matrix with entries $d_{uu} := \sum_{v \sim u} \frac{2\sqrt{d_u d_v}}{d_u + d_v}$, where $v \sim u$ means that v is a neighbor of u, i.e., $uv \in E(G)$. Finally, define the *GA Laplace matrix* $\mathcal{L} := \mathcal{D} - \mathcal{A}$. Note that \mathcal{L} is the classical Laplace matrix for every regular graph.

Denote by A the classical adjacency matrix of a graph and by tr(A) the trace of the matrix A. Since the adjacency matrix A, \mathcal{A} and \mathcal{L} are real symmetric matrices, their eigenvalues are real numbers. Denote by $\lambda_1 \geq \cdots \geq \lambda_n$, $\mu_1 \geq \cdots \geq \mu_n$ and $\eta_1 \geq \cdots \geq \eta_n$ the ordered eigenvalues of A, \mathcal{A} and \mathcal{L} , respectively. It is well known (see, e.g., [8]) that $\sum_{i=1}^n \lambda_i^k$ is equal to the number of closed walks of length k in the graph G.

It is also acknowledged that the second smallest (classical) Laplacian eigenvalue of a graph (its *algebraic connectivity*) is the most important information about its spectrum. This eigenvalue is related to several important graph invariants and provides good bounds on the values of several parameters of graphs which otherwise are hard to compute.

The sum-connectivity matrix S = S(G) of the graph G is defined as the matrix with entries (see [78]):

$$S_{uv} := egin{cases} rac{1}{\sqrt{d_u + d_v}} \,, & ext{if } uv \in E(G), \ 0, & ext{otherwise}. \end{cases}$$

Theorem 3.1. [65, Theorem 2.10] For any graph G,

$$GA(G) \ge \frac{1}{2} tr(\mathcal{A}^2).$$

The equality is attained if and only if G is regular.

[65, Theorem 2.10] was proved for connected graphs, but the argument also works if G is not connected.

Theorem 3.2. [64, Theorem 2.3] Let G be a graph with minimum degree δ . Then

$$GA(G) \ge \delta tr(\mathcal{S}^2),$$

and the equality holds if and only if G is regular.

[64, Theorem 2.3] was proved for connected graphs, but the argument also works if G is not connected.

The following result improves Theorem 3.1.

Theorem 3.3. [65, Theorem 2.13] Let G be a graph with m edges, minimum degree δ and maximum degree Δ . Then

$$GA(G) \ge \sqrt{\frac{1}{2} tr(\mathcal{A}^2) + \frac{4\Delta\delta}{(\Delta+\delta)^2} m(m-1)}.$$

If G is connected, then the equality is attained if and only if G is either regular or biregular.

[65, Theorem 2.13] was proved for connected graphs, but the argument also works if G is not connected.

Given a graph G, denote by N(u) the set of neighbors of the vertex u, and by δ_0 and Δ_0 the integer numbers

$$\delta_0 := \min_{uv \in E(G)} |N(u) \cap N(v)|, \qquad \Delta_0 := \max_{uv \in E(G)} |N(u) \cap N(v)|.$$

It is clear that $0 \le \delta_0 \le \Delta_0 \le \Delta$, where Δ is the maximum degree of *G*.

Theorem 3.4. [65, Theorem 2.14] Let G be a graph with minimum degree δ , maximum degree Δ and $\Delta_0 > 0$. Then

$$GA(G) \ge \frac{\delta^2 tr(\mathcal{A}^3)}{2\,\Delta^2 \Delta_0} \,.$$

The equality is attained if and only if G is regular and $\delta_0 = \Delta_0$.

[65, Theorem 2.14] was proved for connected graphs, but the argument also works if *G* is not connected.

Denote by *A* the adjacency matrix of a graph. Recall that since the adjacency matrix *A* and *A* are real symmetric matrices, their eigenvalues are real numbers. Let us keep the notation $\lambda_1 \ge \cdots \ge \lambda_n$ and $\mu_1 \ge \cdots \ge \mu_n$ for the ordered eigenvalues of *A* and *A*, respectively.

Theorem 3.5. [64, Theorem 3.4] For any connected graph G with n vertices, the following inequality holds

$$GA(G) \ge \frac{\mu_1^2 n}{2(n-1)}.$$

Furthermore, the equality is attained for every complete graph.

Theorem 3.6. [64, Theorem 3.12] For any connected graph G,

$$GA(G) \ge \frac{1}{2} \sum_{j=1}^{n} \lambda_j \mu_{n-j+1}.$$

The next result provides lower bounds of GA involving η_{n-1} .

Theorem 3.7. [65, Theorem 3.12] For any connected graph G with n vertices the following statements hold.

- The geometric-arithmetic index of G satisfies the inequality $GA(G) \ge \frac{1}{2} (n-1)\eta_{n-1}$.
- If G is a bipartite graph with parts X, Y, then $GA(G) \ge \frac{|X||Y|}{n} \eta_{n-1}$.

The energy of the graph G is defined in [26] as $E(G) = \sum_{j=1}^{n} |\lambda_j|$. The geometric-arithmetic energy (GA energy) of the graph G is defined in an analogue way as

$$GAE(G) = \sum_{j=1}^{n} |\mu_j|.$$

Proposition 3.1. [65, Proposition 3.14] Let G be a connected graph with m edges, n vertices, minimum degree δ and maximum degree Δ . Then the following inequalities hold

- $GAE(G) \le \sqrt{2n GA(G)}$,
- $\frac{1}{2n}GAE(G)^2 + \frac{4\Delta\delta}{(\Delta+\delta)^2}m(m-1) \le GA(G)^2$.

Next, some lower bounds of GA that involve some of the matrices introduced above as well as some Zagreb indices.

Theorem 3.8. [64, Theorem 2.5] Let G be a graph with minimum degree δ and maximum degree Δ . Then

$$GA(G) \ge \frac{2\delta\sqrt{\Delta M_2(G)\,tr(\mathcal{S}^2)}}{\Delta^2 + \delta^2}$$

The equality is attained if and only if G is regular.

[64, Theorem 2.5] was proved for connected graphs, but the argument also works if G is not connected.

Theorem 3.9. [64, Theorem 3.14] Let G be a graph with m edges and minimum degree δ . Then

$$GA(G) \ge \frac{\delta^2 tr(\mathcal{A}^3)}{\Delta^2 (M_1(G) - 2m)}$$

The equality is attained if and only if G is regular.

[64, Theorem 3.14] was proved for connected graphs, but the argument also works if G is not connected.

Theorem 3.10. [65, Theorem 2.15] Let G be a graph with m edges and maximum degree Δ . Then

$$GA(G) \ge \frac{1}{2\Delta} tr(\mathcal{A}^4) - \frac{1}{2} \left(M_1(G) - 2m \right).$$

If G is connected, then the equality is attained if and only if G is isomorphic to the complete bipartite graph $K_{\Delta,\Delta}$.

[65, Theorem 2.15] was proved for connected graphs, but the argument also works if G is not connected.

4. Lower bounds of GA(G) involving other topological indices

Next, some lower bounds of *GA* involving the first and second Zagreb indices, and their variable versions.

Theorem 4.1. [63, Theorem 3.7] Let G be a graph with m edges and minimum degree δ . Then

$$GA(G) \ge \frac{2\delta m^2}{M_1(G)}.$$

The equality is attained if and only if G is regular.

[63, Theorem 3.7] was proved for connected graphs, but the argument also works if G is not connected. Note that Theorem 4.1 is a consequence of Proposition 4.2.

Theorem 4.2. [66, Theorem 2] Let G be a graph with minimum degree δ and maximum degree Δ . Then

$$GA(G) \ge \frac{2\delta M_1(G)}{(\Delta + \delta)^2}.$$

The equality is attained if and only if G is regular.

[66, Theorem 2] was proved for connected graphs, but the argument also works if G is not connected.

Theorem 4.2 is improved by the following result.

Theorem 4.3. [25, Theorem 2.1] If G is a graph with maximum degree Δ and minimum degree δ , then

$$GA(G) \ge \frac{1}{2\Delta} M_1(G), \quad \text{if } \delta/\Delta \ge t_0,$$

$$GA(G) \ge \frac{2\sqrt{\Delta\delta}}{(\Delta+\delta)^2} M_1(G), \quad \text{if } \delta/\Delta < t_0,$$
(12)

where t_0 is the unique solution of the equation $t^3 + 5t^2 + 11t - 1 = 0$ in the interval (0, 1). The equality in the first bound is attained if and only if *G* is regular; the equality in the second bound is attained if and only if *G* is a biregular graph.

Theorem 4.4. [62, Theorem 2.4] Let G be a graph with m edges and minimum degree δ . Then

$$GA(G) \ge 2m - \frac{M_1(G)}{2\delta}.$$

The equality is attained if and only if G is regular.

[62, Theorem 2.4] was proved for connected graphs, but the argument also works if *G* is not connected.

Theorem 4.5. [55, Theorem 9] If G is a graph with m edges and minimum degree δ , then

$$GA(G) \ge \frac{2\,\delta^{1/2}m^2}{M_1^{3/2}(G)}$$

and the equality is attained if and only if G is regular.

Theorem 4.6. [25, Theorem 2.7] If $p \ge 2$ and G is a graph with m edges and minimum degree δ , then

$$GA(G) \ge 2 \, \delta^{1/2} m^p M_1^{(2p-1)/(2p-2)}(G)^{1-p}.$$

Corollary 4.1. [46, Corollary 5] We have for any graph G with minimum degree δ , maximum degree Δ and m edges

$$GA(G) \ge \frac{\delta^3 m^2}{\Delta M_2(G)},$$

and the inequality is attained if and only if G is regular.

Theorem 4.7. [63, Theorem 3.10] Let G be a graph with m edges, minimum degree δ and maximum degree Δ . Then

$$GA(G) \ge \frac{2}{\Delta + \delta} \sqrt{\frac{\delta m M_2(G)}{\Delta}}$$

The equality is attained if and only if G is a regular graph.

[63, Theorem 3.10] was proved for connected graphs, but the argument also works if G is not connected. Theorem 4.7 can be obtained by using (1).

Theorem 4.8. [46, Theorem 3] Let G be a graph with m edges and minimum degree δ . Then

$$GA(G) \ge \frac{\delta^2 m^2}{M_2(G)}.$$

The equality is attained if and only if G is regular.

Note that Theorem 4.8 is a consequence of Corollary 4.3.

Theorem 4.9. [66, Theorem 1] Let G be a graph with m edges, minimum degree δ and maximum degree Δ . Then

$$GA(G) \ge \frac{2\sqrt{2\sqrt{\Delta\delta}(\Delta+\delta)mM_2(G)}}{\Delta(\sqrt{\Delta}+\sqrt{\delta})^2} \,.$$

The equality is attained if and only if G is a regular graph.

[66, Theorem 1] was proved for connected graphs, but the argument also works if G is not connected.

The previous result improves the bound given in Theorem 4.7, since, as the author shows in the same paper,

Remark 4.1. For any $0 < \delta \leq \Delta$ we have

$$\frac{\sqrt{2\sqrt{\Delta\delta}(\Delta+\delta)}}{\Delta(\sqrt{\Delta}+\sqrt{\delta}\,)^2} \geq \frac{1}{\Delta+\delta}\sqrt{\frac{\delta}{\Delta}}$$

Theorem 4.10. [62, Theorem 2.3] Let G be a graph with m edges, minimum degree δ and maximum degree Δ . Then

$$GA(G) \ge \frac{\sqrt{(\Delta+\delta)^2 M_2(G) + 4\Delta^3 \delta m(m-1)}}{\Delta(\Delta+\delta)}$$

The equality holds if and only if G is regular.

[62, Theorem 2.3] was proved for connected graphs, but the argument also works if G is not connected.

Proposition 4.1. [64, Proposition 3.11] Let G be a graph with m edges, n vertices and maximum degree Δ . Then

$$GA(G) \ge \frac{\sqrt{n^2 M_2(G) + 4\Delta^2(n-1)m(m-1)}}{n\Delta}.$$

[64, Theorem 3.11] was proved for connected graphs, but the argument also works if *G* is not connected.

Theorem 4.11. [63, Theorem 3.8] Let G be a graph with m edges, minimum degree δ and maximum degree Δ . Then

$$GA(G) \ge \frac{2\delta m}{\Delta^2 + \delta^2} \sqrt{\frac{2\Delta M_2(G)}{M_1(G)}}$$

The equality is attained if and only if G is a regular graph.

[63, Theorem 3.8] was proved for connected graphs, but the argument also works if *G* is not connected.

Note that Theorem 4.11 is a consequence of Corollary 4.9.

In [43], the authors provide correct versions of two statements which had previously appeared in [42]. Both results use the concept of *bidegreed* graph, i.e., a graph whose vertex degree is either Δ or δ with $\Delta > \delta \ge 1$.

Theorem 4.12. [43, Theorem 2.4] Let G be a connected graph of order n with m edges and let p, Δ and δ_1 denote the number of pendant vertices, maximum vertex degree and minimum nonpendant vertex degree of G, respectively. Then,

$$GA(G) \ge \frac{2p\sqrt{\delta_1}}{\Delta+1} + \frac{2\delta_1}{(\Delta^2+\delta_1^2)}\sqrt{(m-p)(M_2(G)-p\Delta)}.$$

The equality holds if and only if G is regular or a bidegreed graph with one vertex set of degree one.

Corollary 4.2. [43, Corollary 2.5] Let T be a tree of order n and with the assumptions in Theorem 4.12. The following inequality holds

$$GA(T) \ge \frac{2p\sqrt{\delta_1}}{\Delta+1} + \frac{2\delta_1}{(\Delta^2+\delta_1^2)}\sqrt{(n-1-p)(M_2(T)-p\Delta)}$$

The equality holds if and only if T is a bidegreed tree.

The following results give lower bounds for GA involving Zagreb indices $M_1(G)$, $M_2(G)$ and $M_2^{-1}(G)$.

Theorem 4.13. [3, Theorem 2.12] If G is a graph, then

$$GA(G) \ge M_2^{-1}(G).$$

The equality is attained if and only if each connected component of G is isomorphic to P_2 .

Although the authors proved [3, Theorem 2.12] for connected graphs, it can be easily checked that the conclusion is also true when that hypothesis is removed. In order to do it, it suffices to apply the theorem to each of the connected components.

Corollary 4.3. [3, Corollary 2.13] If G is a graph with minimum degree δ , then

$$GA(G) \ge \delta^2 M_2^{-1}(G).$$

The equality is attained if and only if G is a regular graph.

Although the authors proved [3, Corollary 2.13] for connected graphs, it can be easily checked that the conclusion is also true when that hypothesis is removed. In order to do it, it suffices to apply the corollary to each of the connected components.

In [3, Corollary 2.13] appears the additional hypothesis $\delta \ge 2$, but Theorem 4.13 shows that the inequality also holds if $\delta = 1$. Note that Corollary 4.3 improves Theorem 4.13 when $\delta \ge 2$.

Theorem 4.14. [66, Theorem 4] Let G be a graph with minimum degree δ and maximum degree Δ . Then

$$GA(G) \ge \frac{2\delta^2 \sqrt{M_2(G)M_2^{-1}(G)}}{\Delta^2 + \delta^2}$$

The equality is attained if and only if G is a regular graph.

[66, Theorem 4] was proved for connected graphs, but the argument also works if G is not connected.

Theorem 4.14 is improved by the following result.

Corollary 4.4. [62, Corollary 2.13] Let G be a graph with minimum degree δ and maximum degree Δ . Then

$$GA(G) \ge \frac{2\delta}{\Delta^2 + \delta^2} \sqrt{\delta \Delta M_2(G) M_2^{-1}(G)}.$$

The equality is attained if and only if G is a regular graph.

[62, Corollary 2.13] was proved for connected graphs, but the argument also works if *G* is not connected.

Theorem 4.15. [66, Theorem 3] Let G be a graph with m edges and maximum degree Δ . Then

$$GA(G) \ge \sqrt{\frac{m^3}{\Delta^2 M_2^{-1}(G)}} \,.$$

The equality is attained if and only if G is regular.

[66, Theorem 3] was proved for connected graphs, but the argument also works if G is not connected.

Theorem 4.16. [62, Theorem 2.10] Let G be a graph with m edges, minimum degree δ and maximum degree Δ . Then

$$GA(G) \ge \frac{2\Delta\delta^2}{\Delta^2 + \delta^2} \sqrt{mM_2^{-1}(G)}.$$

The equality is attained if and only if G is a regular graph.

[62, Theorem 2.10] was proved for connected graphs, but the argument also works if G is not connected.

Theorem 4.17. [62, Theorem 2.11] Let G be a graph with minimum degree δ and maximum degree Δ . Then

$$GA(G) \ge \frac{4\Delta^2 \delta^2 \sqrt{2\delta M_1(G)M_2^{-1}(G)}}{(\Delta^2 + \delta^2)(\delta + \Delta)^2},$$

The equality is attained if and only if G is a regular graph.

[62, Theorem 2.11] was proved for connected graphs, but the argument also works if G is not connected.

Next, some lower bounds of GA involving other variable Zagreb indices.

Theorem 4.18. [46, Theorem 1] Let G be a graph with m edges, minimum degree δ , maximum degree Δ and let $\alpha \in \mathbb{R}$. Then

$$GA(G) \ge \frac{c_{\alpha}m^2}{M_2^{\alpha}(G)},$$

with

$$c_{\alpha} := \begin{cases} \delta^{2\alpha+1} \Delta^{-1}, & \text{if } \alpha \ge -1/2, \\ \Delta^{2\alpha}, & \text{if } \alpha \le -1/2. \end{cases}$$

The equality is attained for some fixed α if and only if G is regular.

Theorem 4.19. [62, Theorem 2.6] Let G be a graph with minimum degree δ , maximum degree Δ and $\alpha \in \mathbb{R} \setminus \{0\}$. Then the following statements hold.

(a) If $\alpha \leq -1/2$, then $GA(G) \geq \delta^{-2\alpha} M_2^{\alpha}(G)$.

(b) If
$$\alpha \geq -1/2$$
, then $GA(G) \geq \delta \Delta^{-2\alpha-1} M_2^{\alpha}(G)$

Both equalities are attained if and only if G is a regular graph.

[62, Theorem 2.6] was proved for connected graphs, but the argument also works if G is not connected.

Theorem 4.20. [62, Theorem 2.8] Let G be a graph with minimum degree δ , maximum degree Δ and $\alpha \in \mathbb{R} \setminus \{0\}$. Then the following statements hold.

(a) If $\alpha \leq 1/2$, then $GA(G) \geq \delta^{-2\alpha+1} \Delta^{-1} M_2^{\alpha}(G)$.

(b) If $\alpha \geq 1/2$, then $GA(G) \geq \Delta^{-2\alpha} M_2^{\alpha}(G)$.

Both equalities are attained if and only if G is a regular graph.

[62, Theorem 2.8] was proved for connected graphs, but the argument also works if G is not connected.

The following result improves Theorems 4.13, 4.19 and 4.20, and generalizes Corollary 4.3.

Theorem 4.21. [25, Theorem 2.9] Let G be a graph with maximum degree Δ and minimum degree δ , and $\alpha \in \mathbb{R}$. Then

$$GA(G) \ge \delta^{-2\alpha} M_2^{\alpha}(G), \quad \text{if } \alpha \le -1/2,$$

$$GA(G) \ge \min\left\{\delta^{-2\alpha}, \frac{2}{(\Delta\delta)^{\alpha-1/2}(\Delta+\delta)}\right\} M_2^{\alpha}(G), \quad \text{if } -1/2 < \alpha \le 0,$$

$$GA(G) \ge \min\left\{\Delta^{-2\alpha}, \frac{2}{(\Delta\delta)^{\alpha-1/2}(\Delta+\delta)}\right\} M_2^{\alpha}(G), \quad \text{if } 0 < \alpha < 1/2,$$

$$GA(G) \ge \Delta^{-2\alpha} M_2^{\alpha}(G), \quad \text{if } \alpha \ge 1/2.$$

Every equality is attained for every regular graph G. Furthermore, the equalities in the first and fourth cases are attained if and only if G is regular.

Corollary 4.4 is generalized by the following result.

Theorem 4.22. [62, Theorem 2.12] Let G be a graph with minimum degree δ and maximum degree Δ , and let $\alpha > 0$. Then

$$GA(G) \ge k_{\alpha} \sqrt{M_2^{\alpha}(G)M_2^{-\alpha}(G)},$$

with

$$k_{\alpha} := \begin{cases} \frac{2\Delta^{1/2}\delta^{3/2}}{\Delta^{2}+\delta^{2}}, & \text{if } 0 < \alpha \le 1, \\\\ \frac{2\Delta^{\alpha-1/2}\delta^{\alpha+1/2}}{\Delta^{2\alpha}+\delta^{2\alpha}}, & \text{if } \alpha \ge 1. \end{cases}$$

The equality is attained if and only if G is a regular graph.

[62, Theorem 2.12] was proved for connected graphs, but the argument also works if G is not connected.

Theorem 4.23. [25, Theorem 2.4] If G is a graph with m edges and minimum degree δ , then

$$GA(G) \ge m - \frac{M_1(G) - 2M_2^{1/2}(G)}{2\delta}$$

and the equality is attained if and only if G is regular.

Multiplicative versions of Zagreb indices were introduced in [69] and defined as

$$\Pi_1(G) = \prod_{u \in V(G)} d_u^2 \qquad \text{and} \qquad \Pi_2(G) = \prod_{uv \in E(G)} d_u d_v.$$

One year later, the multiplicative sum Zagreb index, Π_1^* , was introduced in [15] and defined as

$$\Pi_1^*(G) = \prod_{uv \in E(G)} (d_u + d_v).$$

Corollary 4.5. [49, Corollary 1] Let G be a connected graph with m edges. Then

$$GA(G) \ge \sqrt{\frac{1}{2}tr(A^2) + m(m-1)\frac{4(\Pi_2)^{1/m}}{(\Pi_1^*)^{2/m}}}.$$

Next, some lower bounds of *GA* involving the Randić index.

Theorem 4.24. [63, Theorem 3.4] Let G be a graph with m edges and maximum degree Δ . Then

$$GA(G) \ge \frac{m^2}{\Delta R(G)}$$
,

and the equality holds if and only if G is regular.

[63, Theorem 3.4] was proved for connected graphs, but the argument also works if G is not connected.

Corollary 4.6. [3, Corollary 2.5] If G is a connected graph with $n \ge 3$ vertices, then

$$GA(G) \ge \sqrt{4/3} R(G).$$

The equality is attained if and only if $G \cong P_3$.

The following result improves Corollary 4.6 when $\delta \geq 2$.

Theorem 4.25. [66, Theorem 6] Let G be a graph with minimum degree δ . Then

$$GA(G) \ge \delta R(G)$$

and the equality holds if and only if G is regular.

Although the authors proved [66, Theorem 6] for connected graphs, it can be easily checked that the conclusion is also true when that hypothesis is removed. In order to do it, it suffices to apply the theorem to each of the connected components.

Theorem 4.26. [62, Theorem 2.4] Let G be a graph with m edges and maximum degree Δ . Then

$$GA(G) \ge 2m - \Delta R(G),$$

and the equality holds if and only if G is regular.

[62, Theorem 2.4] was proved for connected graphs, but the argument also works if *G* is not connected.

In 1987 [18], Fajtlowicz introduced the *harmonic index* H(G) of a graph G, defined as

$$H(G) = \sum_{uv \in E(G)} \frac{2}{d_u + d_v}$$

Although this quantity was first mentioned in a mathematical paper in 1987, it did not attract the attention of scholars until quite recently. In the last few years, a remarkably large number of studies of the properties of the harmonic index have appeared (see, e.g., [12,41,76,77,80]). The chemical applicability of the harmonic index was also recently investigated [23,33]. The harmonic index has reasonably good correlation abilities; in fact, it gives similar correlations with physical and chemical properties compared with the well-known Randić index.

Proposition 4.2. [63, Proposition 3.9], [3, Corollary 2.7] For any graph G with minimum degree δ

$$GA(G) \ge \delta H(G),$$

and the equality is attained if and only if G is regular.

Although the authors proved this proposition for connected graphs, it can be easily checked that the conclusion is also true when that hypothesis is removed. In order to do it, it suffices to apply the proposition to each of the connected components.

Note that Proposition 4.2 can be deduced from Theorem 3.2, since $tr(S^2) = H(G)$.

Theorem 4.27. [25, Theorem 2.3] If G is a graph with m edges, maximum degree Δ and minimum degree δ , then

$$GA(G) \ge \frac{m^2}{\Delta H(G)}, \quad \text{if } \delta/\Delta \ge t_0,$$

$$GA(G) \ge \frac{4\sqrt{\Delta\delta} m^2}{(\Delta+\delta)^2 H(G)}, \quad \text{if } \delta/\Delta < t_0,$$
(13)

where t_0 is the unique solution of the equation $t^3 + 5t^2 + 11t - 1 = 0$ in the interval (0, 1). The equality in the first bound is attained if and only if *G* is regular; the equality in the second bound is attained if and only if *G* is a biregular graph.

In what follows, and for the sake of simplicity in the notation, we will follow the usual convention and denote by $\chi(G)$ the *sum-connectivity* index, i.e., $\chi_{-1/2}(G)$.

Theorem 4.28. [63, Theorem 3.11] Let G be a graph with m edges and minimum degree δ . Then

$$GA(G) \ge \frac{2\delta \chi(G)^2}{m}$$

and the equality holds if and only if G is regular.

[63, Theorem 3.11] was proved for connected graphs, but the argument also works if G is not connected.

Theorem 4.29. [3, Theorem 2.1] If G is a graph, then

$$GA(G) \ge \sqrt{2} \chi(G).$$

The bound is attained if and only if each connected component of G is isomorphic to P_2 .

Although the authors proved [3, Theorem 2.1] for connected graphs, it can be easily checked that the conclusion is also true when that hypothesis is removed. In order to do it, it suffices to apply the theorem to each of the connected components.

If the minimum degree of G is at least 2, then the bound in Theorem 4.29 can be improved:

Corollary 4.7. [3, Corollary 2.2] If G is a graph with minimum degree δ , then

$$GA(G) \ge \sqrt{2\delta} \chi(G)$$

with equality if and only if G is a δ -regular graph.

Although the authors proved [3, Corollary 2.2] for connected graphs, it can be easily checked that the conclusion is also true when that hypothesis is removed. In order to do it, it suffices to apply the corollary to each of the connected components.

In [3, Corollary 2.2] appears the additional hypothesis $\delta \ge 2$, but Theorem 4.29 shows that the inequality also holds if $\delta = 1$.

Theorem 4.30. [25, Theorem 2.6] If $\alpha > 0$ and G is a graph with m edges and minimum degree δ , then

$$GA(G) \ge 2\delta m^{(\alpha+1)/\alpha} \chi_{\alpha}(G)^{-1/\alpha}$$

and the equality is attained if and only if G is regular.

The *atom-bond connectivity* index of a graph G, abbreviated as ABC(G), was introduced in [17] and defined as

$$ABC(G) = \sum_{uv \in E(G)} \sqrt{\frac{d_u + d_v - 2}{d_u d_v}}$$

The *ABC* index provides a good model for the stability of linear and branched alkanes as well as the strain energy of cycloalkanes (see [16, 17]).

Recall that a *chemical graph* is a connected graph with maximum degree at most four and that the *line graph* $\mathcal{L}(G)$ of G is a graph whose vertices are the edges of G, and such that two vertices are incident if and only if they have a common end vertex in G.

Theorem 4.31. [61, Theorem 2] Let G be a graph which is the line of a chemical graph with at least 3 vertices. Then

Theorem 4.32. [61, Theorem 3] Let G be a graph with maximum degree Δ and minimum degree $\delta \geq 2$. If $\Delta - \delta \leq (2\delta - 1)^2$, then

Although the authors proved [61, Theorem 3] for connected graphs, it can be easily checked that the conclusion is also true when that hypothesis is removed. In order to do it, it suffices to apply the theorem to each of the connected components.

Theorem 4.33. [61, Theorem 4] Let G be a graph with minimum degree $\delta \ge 2$ and $|d_u - d_v| \le (2\delta - 1)^2$ for all edges $uv \in E(G)$. Then

Although the authors proved [61, Theorem 4] for connected graphs, it can be easily checked that the conclusion is also true when that hypothesis is removed. In order to do it, it suffices to apply the theorem to each of the connected components.

Theorem 4.34. [61, Theorem 5] Let G be a graph with minimum degree $\delta \ge 2$ and $|d_u - d_v| \le (2k - 1)^2$ for all edges $uv \in E(G)$ where $k = \min\{d_u, d_v\}$. Then

Although the authors proved [61, Theorem 5] for connected graphs, it can be easily checked that the conclusion is also true when that hypothesis is removed. In order to do it, it suffices to apply the theorem to each of the connected components.

Theorem 4.35. [3, Theorem 2.8] If G is a connected graph having $n \ge 3$ vertices with minimum degree $\delta \ge 2$, then

$$GA(G) \ge \frac{4\sqrt{n-1}}{n+1} ABC(G).$$

The bound is attained if and only if $G \cong C_3$.

In the same paper, where Zagreb indices were introduced, the forgotten topological index (or F-index) is defined as

$$F(G) = \sum_{u \in V(G)} d_u^3 = \sum_{uv \in E(G)} (d_u^2 + d_v^2).$$

Both the forgotten topological index and the first Zagreb index were employed in the formulas for total π -electron energy in [34], as a measure of branching extent of the carbon-atom skeleton of the underlying molecule. However, this index never got attention except recently, when Furtula and Gutman in [22] established some basic properties of the F-index and showed that its predictive ability is almost similar to that of first Zagreb index and for the entropy and acetic factor, both of them yield correlation coefficients greater than 0.95. Besides, [22] pointed out the importance of the F-index: it can be used to obtain a high accuracy of the prediction of logarithm of the octanol-water partition coefficient (see also [1]).

Theorem 4.36. [46, Theorem 5] Let G be a graph with minimum degree δ and m edges. Then

$$GA(G) \ge 2m - \frac{F(G)}{2\delta^2}$$

and the equality is attained if and only if G is regular.

Note that Theorem 4.36 is a consequence of Theorem 4.4.

The modified Narumi-Katayama index

$$NK^*(G) = \prod_{u \in V(G)} d_u^{d_u} = \prod_{uv \in E(G)} d_u d_v$$

was introduced in [24], inspired in the Narumi-Katayama index defined in [51]. Note that $NK^*(G) = \Pi_2(G)$.

Theorem 4.37. [63, Theorem 3.13] Let G be a graph with m edges and maximum degree Δ . Then

$$GA(G) \geq \frac{m}{\Delta} NK^*(G)^{1/(2m)}$$

and the equality is attained if and only if G is regular.

[63, Theorem 3.13] was proved for connected graphs, but the argument also works if G is not connected.

Corollary 4.8. [62, Corollary 2.15] Let G be a graph with m edges, minimum degree δ and maximum degree Δ and let $\alpha \in \mathbb{R} \setminus \{0\}$

$$GA(G) \ge \delta^{-2\alpha} m NK^*(G)^{\alpha/m}, \quad \text{if } \alpha \le -1/2,$$

$$GA(G) \ge \delta \Delta^{-2\alpha-1} m NK^*(G)^{\alpha/m}, \quad \text{if } \alpha \ge -1/2.$$

and the equalities are attained if and only if G is regular.

[62, Corollary 2.15] was proved for connected graphs, but the argument also works if G is not connected.

In [62, Theorem 2.14] appears the inequality $M_2^{\alpha}(G) \ge m NK^*(G)^{\alpha/m}$, with equality for every regular graph; it was proved just for connected graphs, but the argument also works if G is not connected. Using this result and Theorem 4.21, we can obtain the following new inequalities that improve Corollary 4.8 and generalize Theorem 4.37.

Proposition 4.3. Let G be a graph with maximum degree Δ and minimum degree δ , and $\alpha \in \mathbb{R}$. Then

$$GA(G) \ge \delta^{-2\alpha} m NK^*(G)^{\alpha/m}, \quad \text{if } \alpha \le -1/2,$$

$$GA(G) \ge \min\left\{\delta^{-2\alpha}, \frac{2}{(\Delta\delta)^{\alpha-1/2}(\Delta+\delta)}\right\} m NK^*(G)^{\alpha/m}, \quad \text{if } -1/2 < \alpha \le 0,$$

$$GA(G) \ge \min\left\{\Delta^{-2\alpha}, \frac{2}{(\Delta\delta)^{\alpha-1/2}(\Delta+\delta)}\right\} m NK^*(G)^{\alpha/m}, \quad \text{if } 0 < \alpha < 1/2,$$

$$GA(G) \ge \Delta^{-2\alpha} m NK^*(G)^{\alpha/m}, \quad \text{if } \alpha \ge 1/2.$$

Every equality is attained for every regular graph G. Furthermore, the equalities in the first and fourth cases are attained if and only if G is regular.

A family of degree-based structure-descriptors, named *Adriatic indices*, was put forward in [72, 74]. Twenty of them were selected as significant predictors. Among them, the *inverse sum indeg* index, *ISI*, was singled out in [72] as a significant indicator of total surface area of octane isomers. This index is defined as

$$ISI(G) = \sum_{uv \in E(G)} \frac{d_u \, d_v}{d_u + d_v} = \sum_{uv \in E(G)} \frac{1}{\frac{1}{d_u} + \frac{1}{d_v}}.$$

Theorem 4.38. [31, Theorem 1] Let G be a graph with maximum degree Δ . Then

$$GA(G) \ge \frac{2}{\Delta} ISI(G) \,,$$

and the equality is attained if and only if G is regular.

The Albertson index is defined in [2] as

$$Alb(G) = \sum_{uv \in E(G)} |d_u - d_v|.$$

This index is also known as *third Zagreb index* (see [19]) and *misbalance deg index* (see [72, 74]). It is a significant predictor of standard enthalpy of vaporization for octane isomers (see [74]) and it is widely used as a measure of non-regularity of a graph.

Theorem 4.39. [25, Theorem 2.10] If G is a graph with maximum degree Δ and minimum degree δ , then

$$GA(G) \ge \frac{2\sqrt{\Delta\delta}}{\Delta^2 - \delta^2} Alb(G) \,,$$

and the equality is attained if and only if G is a regular or biregular graph.

The global cyclicity index was introduced in [36] and defined as

$$C(G) = \sum_{uv \in E(G)} \frac{1}{R_{uv}} - m,$$

where m is the number of edges of G and R_{uv} denotes the effective resistance between the vertices u and v, that is, the voltage drop between vertices u and v when a battery is installed between those two vertices such that a unit current flows between them.

Proposition 4.4. [54, Proposition 2] Let G be a graph with m edges, minimum degree δ and maximum degree Δ . Then

$$GA(G) \ge \frac{2\delta}{\Delta(\delta+1)} \left(C(G) + m\right)$$

Next, some lower bounds of GA that involve more than one topological index.

Corollary 4.9. [64, Corollary 2.6] Let G be a graph with minimum degree δ and maximum degree Δ . Then

$$GA(G) \geq \frac{2\delta \sqrt{\Delta M_2(G) \, H(G)}}{\Delta^2 + \delta^2}$$

The equality is attained if and only if G is regular.

[64, Corollary 2.6] was proved for connected graphs, but the argument also works if G is not connected. Corollary 4.9 can be deduced from Theorem 3.8, since $tr(S^2) = H(G)$.

Theorem 4.40. [46, Theorem 2] Let G be a graph with minimum degree δ and maximum degree Δ . Then

$$GA(G) \ge \frac{4\,\Delta\delta\sqrt{M_2(G)\,\chi_{-2}(G)}}{\Delta^2 + \delta^2}\,.$$

The equality is attained if and only if G is regular.

Theorem 4.41. [46, Theorem 4] For any graph G

$$GA(G) \ge \frac{H(G)^2}{M_2^{-1}(G)},$$

and the equality is attained if and only if G is regular.

Theorem 4.42. [63, Theorem 3.12] For any graph G,

$$GA(G) \ge \frac{2 \chi(G)^2}{R(G)}$$
.

If G is connected, then the equality is attained if and only if G is regular or biregular.

[63, Theorem 3.12] was proved for connected graphs, but the argument also works if G is not connected.

Theorem 4.43. [25, Theorem 2.13] If G is a graph with maximum degree Δ and minimum degree δ , then

$$GA(G) \ge \frac{4\Delta^2 \delta^2 \sqrt{(F(G) + 2M_2(G))M_2^{-1}(G)}}{(\Delta + \delta)^2 (\Delta^2 + \delta^2)},$$

and the equality is attained if and only if G is regular.

In [21], the symmetric division deg index is defined as

$$SDD(G) = \sum_{uv \in E(G)} \left(\frac{d_u}{d_v} + \frac{d_v}{d_u} \right)$$

It turns out to be useful predicting some physico-chemical properties of molecules (in particular, for PCB's); besides, the authors think that this definition fills the gap among vertex-degree-based indices corresponding to the fourth basic arithmetic operation: division. They think that this fact could stimulate the mathematically oriented researchers to pay some attention to this graph invariant.

Theorem 4.44. [21, Theorem 7] Let G be a graph of order n with m edges. Then,

$$GA(G) \ge \frac{2m^2}{SDD(G)}$$

with equality holding if and only if each connected component of G is a regular graph.

Next, some lower bounds of *GA* in the form of chains of inequalities among different indices of a graph.

Corollary 4.10. [3, Corollary 2.9] If G is a connected graph with n vertices and minimum degree $\delta \geq 2$, then

$$H(G) \le R(G) \le \chi(G) < ABC(G) \le \frac{n+1}{4\sqrt{n-1}} GA(G)$$

with the first equality if and only if G is a regular graph, the second equality if and only if G is a cycle and last equality if and only if $G \cong C_3$.

Denote by T^* the tree with eight vertices, obtained by joining the central vertices of two copies of the star graph $K_{1,3}$ by an edge.

Corollary 4.11. [3, Corollary 2.10] If G is a connected graph with minimum degree δ and maximum degree Δ satisfying at least one of the following properties:

- i) G is a chemical graph such that $G \ncong K_{1,4}, T^*$,
- ii) $\Delta \delta \leq 3$ and $G \ncong K_{1,4}, T^*$,
- iii) $\delta \geq 2$ and $\Delta \delta \leq (2\delta 1)^2$, then

 $H(G) \le R(G) \le \chi(G) < ABC(G) < GA(G).$

5. Lower bounds of GA(G) in terms of subgraphs of G

We say that a family of subgraphs $\{G_1, \ldots, G_r\}$ of G is a *decomposition* of G if $G_1 \cup \cdots \cup G_r = G$ and $G_i \cap G_j$ is either empty or a vertex for every $i, j \in \{1, \ldots, r\}, i \neq j$. The subgraphs G_1, \ldots, G_r are usually called *primary subgraphs* of the decomposition.

If $v \in V(G)$, then denote by $N_G(v)$ or N(v) the set of neighbors of v, i.e.,

$$N_G(v) = N(v) = \{ u \in V(G) \mid uv \in E(G) \}.$$

Given a decomposition $\{G_1, \ldots, G_r\}$ of G, denote by \mathcal{W} the set of vertices v in G belonging at least to two different G_i 's. Given a vertex $v \in \mathcal{W}$, denote by G_{i_1}, \ldots, G_{i_k} the set of primary subgraphs containing v and by d_{i_j} the number of neighbors of v in G_{i_j} (then $d_v = d_{i_1} + \cdots + d_{i_k}$). If $v \in \mathcal{W}$, then define W(v) as

$$W(v) = \sum_{u \in N_G(v) \setminus \mathcal{W}} \frac{\sqrt{d_u d_v}}{\frac{1}{2}(d_u + d_v)} - \sum_{j=1}^k \sum_{u \in N_{G_{i_j}}(v) \setminus \mathcal{W}} \frac{\sqrt{d_u d_{i_j}}}{\frac{1}{2}(d_u + d_{i_j})}$$

Denote by Z the set of edges in G with both endpoints in W. If $e = uv \in Z$, then $e \in G_i$ for some i, and we denote by d_u^*, d_v^* the degrees of u, v in G_i . If $e = uv \in Z$, then define Z(e) as

$$Z(e) = \frac{\sqrt{d_u d_v}}{\frac{1}{2}(d_u + d_v)} - \frac{\sqrt{d_u^* d_v^*}}{\frac{1}{2}(d_u^* + d_v^*)}$$

Proposition 5.1. [35, Proposition 2.6] Let $\{G_1, \ldots, G_r\}$ be a decomposition of the connected graph G. If $d_v \leq d_u$ for every $v \in W$ and $u \in N_G(v) \setminus W$, then

$$GA(G) \ge \sum_{i=1}^{r} GA(G_i) - \operatorname{card} \mathcal{Z}.$$

Furthermore, if every edge in Z is maximal, then

$$GA(G) \ge \sum_{i=1}^{r} GA(G_i).$$

Corollary 5.1. [35, Corollary 2.7] Let $\{G_1, \ldots, G_r\}$ be a decomposition of the connected graph G, whose minimum degree is δ . If $d_v = \delta$ for every $v \in W$, then

$$GA(G) \ge \sum_{i=1}^{r} GA(G_i) - card \mathcal{Z}.$$

Furthermore, if every edge in Z is maximal, then

$$GA(G) \ge \sum_{i=1}^{r} GA(G_i).$$

In [7] the authors find lower bounds for the geometric-arithmetic index in terms of this same graph invariant of the subgraph obtained when one edge (with certain properties) is removed.

Theorem 5.1. [7, Theorem 3.4] For an edge $e = v_i v_j$ of a graph, let $d_r = \max\{d_k | v_i v_k \in E(G)\}$ and $d_s = \max\{d_l | v_j v_l \in E(G)\}$. If one of the following conditions is satisfied,

(i) $\max\{d_i/d_r, d_j/d_s\} \le 1 \text{ or }$

(*ii*) $\max\{d_i/d_j, d_j/d_i\} \le \min\{d_i/d_r, d_j/d_s\},$ then GA(G) > GA(G-e).

Let e = uv be an edge of a graph G. We define the *weight* of e = uv as $w(e) = \frac{2\sqrt{d_u d_v}}{d_u + d_v}$, and we say that e is an edge of maximum weight if $w(e) \ge w(f)$ for any $f \in E(G)$. Define $N(e) = \{f \in E(G) \mid f \text{ is adjacent to } e\}$ and $N[e] = N(e) \cup \{e\}$.

Corollary 5.2. [7, Corollary 3.5] For an edge e of G with maximum weight in N[e],

$$GA(G) > GA(G-e)$$

Corollary 5.3. [7, Corollary 3.6] For an edge *e* of *G* with maximum weight in *G*,

$$GA(G) > GA(G-e)$$

6. Lower bounds of GA(G) for line graphs

Recall that the *line graph* $\mathcal{L}(G)$ of G is a graph whose vertices are the edges of G and such that two vertices are incident if and only if they have a common end vertex in G.

Along this section, by a non-trivial graph we mean a graph such that each connected connected component has at least two edges.

The proof of the following result uses (1).

Proposition 6.1. [55, Proposition 6] If G is a non-trivial graph with m edges, maximum degree Δ and minimum degree δ , then

$$GA(\mathcal{L}(G)) \ge \frac{(M_1(G) - 2m)\sqrt{(\Delta - 1)(\delta - 1)}}{\Delta + \delta - 2}$$

The equality is attained if and only if G is regular.

In 1956, Nordhaus and Gaddum [53] gave bounds involving the sum of the chromatic number of a graph and its complement. Motivated by these results, Das obtains in [9] analogous conclusions for the geometric-arithmetic index of a graph and its complement. The next theorem is also a Nordhaus-Gaddum-type result for the geometric-arithmetic index of a graph and its line graph.

Corollary 6.1. [55, Corollary 2] If G is a non-trivial graph with maximum degree Δ and minimum degree δ , then

$$GA(G) + GA(\mathcal{L}(G)) \ge \frac{M_1(G)\sqrt{(\Delta - 1)(\delta - 1)}}{\Delta + \delta - 2}$$

and the equality is attained if and only if G is regular.

Theorem 6.1. [55, Theorem 3] If G is a non-trivial graph with maximum degree Δ and minimum degree δ , then

$$GA(\mathcal{L}(G)) \ge \min\left\{\frac{3}{4\sqrt{2}}, \frac{2\sqrt{(2\Delta-2)\max\{2\delta-2,1\}}}{2\Delta-2+\max\{2\delta-2,1\}}\right\}GA(G).$$

The equality is attained for the path graph P_3 .

Next, the bound in Theorem 6.1 is improved for a special class of graphs.

Theorem 6.2. [55, Theorem 4] Let G be a non-trivial graph such that each connected component of G is regular or biregular and it is not isomorphic to P_3 . Then $GA(\mathcal{L}(G)) \ge GA(G)$, and the equality is attained for every union of cycle graphs.

The following result is a consequence of the first inequality in (2).

Corollary 6.2. [55, Corollary 4] If G is a non-trivial connected graph with m edges, then

$$GA(\mathcal{L}(G)) \ge \frac{2(m-1)^{3/2}}{m} \,.$$

Theorem 6.3. [55, Theorem 6] Let G be a non-trivial graph with m edges such that each connected component of G is not isomorphic to a path graph P_n with $n \leq 6$. Then

$$GA(\mathcal{L}(G)) \ge 2\sqrt{m-1}$$

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