

The distance spectrum of the subdivision vertex join and subdivision edge join of two regular graphs

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(Received: 7 October 2018. Received in revised form: 24 November 2018. Accepted: 27 November 2018. Published online: 5 January 2019.)

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Abstract

Let D be the distance matrix of a connected graph G . The D -eigenvalues $\mu_1, \mu_2, \dots, \mu_p$ of G are the eigenvalues of D and form the distance spectrum or D -spectrum of G . The subdivision graph $S(G)$ of a graph G is obtained from G by inserting a new vertex of degree 2 in every edge of G ; we denote the set of such new vertices by $I(G)$. The subdivision-vertex join of two vertex disjoint graphs G_1 and G_2 , denoted by $G_1 \dot{\vee} G_2$, is the graph obtained from $S(G_1)$ and G_2 by joining each vertex of $V(G_1)$ with every vertex of $V(G_2)$. The subdivision-edge join of two vertex disjoint graphs G_1 and G_2 , denoted by $G_1 \underline{\vee} G_2$, is the graph obtained from $S(G_1)$ and G_2 by joining each vertex of $I(G_1)$ with every vertex of $V(G_2)$. In this paper, we find the distance spectrum of $G_1 \dot{\vee} G_2$ and $G_1 \underline{\vee} G_2$, when G_1 and G_2 are regular graphs. Thus, we add a new class of graphs to the classes of those graphs whose distance spectrum is known.

Keywords: distance matrix; distance spectrum; subdivision-vertex join; subdivision-edge join.

2010 Mathematics Subject Classification: 05C12, 05C50.

1. Introduction

Spectral theory of graphs is an emerging area of algebraic graph theory which covers the spectra of the matrices associated with graphs. From the fundamental work of Hückel [7] on the eigenvalues of molecular graphs, the study of several graph polynomials and the associated spectra have been the topic of various papers in the last few years. The characteristic polynomial of the adjacency matrix and its spectrum have been calculated for a variety of graphs. On the other hand, the characteristic polynomial of the distance matrix was not studied in depth; even though the distance matrix reflects the structure of a graph more clearly than the adjacency matrix. A significant result involving the distance spectrum was appeared in a seminal work of Graham and Pollack [5] in 1971, which is related to the number of negative eigenvalues of the distance matrix.

Since all the off-diagonal entries of the distance matrix of a connected graph are nonzero, the study of the characteristic polynomial (and consequently the study of the spectrum) seems to be computationally more difficult than that of the adjacency matrix and, in general, there are no simple analytical or compact solutions except those for a few trees [4]. This may be a reason which accounts for the detailed study of this spectrum only for trees.

Let G be a connected graph with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$. The distance matrix $D = D(G)$ of G is the matrix whose (i, j) -entry is equal to the distance $d_G(v_i, v_j)$, that is, the length of the shortest path [2] between the vertices v_i and v_j of G . The eigenvalues of $D(G)$ are said to be the D -eigenvalues of G and form the distance spectrum (D -spectrum for short) of G , denoted by $\text{spec}_D(G)$.

The distance spectrum of cycles were discussed in [6] and that of complete and complete bipartite graphs can be seen in [17]. The distance spectrum of an n -vertex path P_n and the first eigenvector of the distance matrix were obtained in [19]. In [21], the authors determined the distance spectrum of the graphs obtained from regular graphs by applying some join related operations and this result was generalized Stevanović [20]. The D -spectrum of the cartesian product of two distance regular graphs and also the D -spectrum of the lexicographic product of a graph with a regular graph were presented in [10]. Construction of a graph, considered in [22], was generalized in [11] and its distance spectrum were studied there. Further detail about the D -spectrum of graphs can be found in [1, 8, 12–16].

The subdivision graph $S(G)$ of a graph G is obtained from G by inserting a new vertex of degree 2 in every edge of G ; we denote the set of such new vertices by $I(G)$. In [9], the following graph operations, based on subdivision graph were introduced.

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Definition 1.1. The subdivision-vertex join of two vertex disjoint graphs G_1 and G_2 , denoted by $G_1 \dot{\vee} G_2$, is the graph obtained from $S(G_1)$ and G_2 by joining each vertex of $V(G_1)$ with every vertex of $V(G_2)$.

Definition 1.2. The subdivision-edge join of two vertex disjoint graphs G_1 and G_2 , denoted by $G_1 \dot{\cup} G_2$, is the graph obtained from $S(G_1)$ and G_2 by joining each vertex of $I(G_1)$ with every vertex of $V(G_2)$.

The adjacency spectrum of the graphs $G_1 \dot{\vee} G_2$ and $G_1 \dot{\cup} G_2$ were obtained in [9, 18], whereas the distance spectrum of these two graphs have not yet been studied. In this paper, we obtain the distance spectrum of the graphs $G_1 \dot{\vee} G_2$ and $G_1 \dot{\cup} G_2$ when G_1 and G_2 are regular graphs.

All the graphs considered in this paper are simple. We follow [3] for the spectral graph theoretic terminology.

The discussions in the subsequent sections are based upon the following lemmas:

Lemma 1.1. [3] Let G be an r -regular graph with an adjacency matrix A and an incidence matrix R . Let $L(G)$ be the line graph of G . Then $RR^T = A + rI$, $R^T R = A(L(G)) + 2I$. Also, if J is an all-one matrix of appropriate order then $JR = 2J = R^T J$ and $JR^T = rJ = RJ$

Lemma 1.2. [3] Let G be an r -regular (p, q) graph with $\text{spec}(G) = \{r, \lambda_2, \dots, \lambda_p\}$. Then

$$\text{spec}(L(G)) = \begin{pmatrix} 2r-2 & \lambda_2+r-2 & \dots & \lambda_p+r-2 & -2 \\ 1 & 1 & \dots & 1 & q-p \end{pmatrix}.$$

Also, Z is an eigenvector corresponding to the eigenvalue -2 if and only if $RZ = 0$ where R is the incidence matrix of G .

2. The distance spectrum of $G_1 \dot{\vee} G_2$

In this section, we obtain the distance spectrum of $G_1 \dot{\vee} G_2$ when G_1 and G_2 are two regular graphs.

Theorem 2.1. Let G_i be an r_i regular graph on p_i vertices and q_i edges with the adjacency matrix A_i and adjacency spectrum $\{r_i, \lambda_{i2}, \lambda_{i3}, \dots, \lambda_{ip_i}\}$, $i = 1, 2$. Then, the distance spectrum of $G_1 \dot{\vee} G_2$ consists of the following numbers: $-2(\lambda_{1j} + r_1 + 1)$, $j = 2, 3, \dots, p_1$; 0 of multiplicity $q_1 - 1$; $-(\lambda_{2j} + 2)$, $j = 2, 3, \dots, p_2$, along with the three roots of the equation

$$\begin{aligned} &x^3 - (2p_1 + 2p_2 + 4q_1 - 4r_1 - r_2 - 4)x^2 + (4p_1 + 4p_2 - 3p_1p_2 + 4q_1 + p_1q_1 - 4p_2q_1 - 8r_1 + 2p_1r_1 + 8p_2r_1 - 2r_2 \\ &+ 2p_1r_2 + 4q_1r_2 - 4r_1r_2 - 4)x + 2p_1r_1r_2 + p_1q_1r_2 - 4q_1r_2 - 4p_1p_2r_1 + 4p_1r_1 - 2p_1p_2q_1 + 8p_2q_1 + 2p_1q_1 - 8q_1 = 0. \end{aligned}$$

Proof. Given that G_1 and G_2 are regular graphs with regularity r_1 and r_2 respectively. Let R be the incidence matrix of G_1 and B be the adjacency matrix of $L(G_1)$. Then, by a proper ordering of the vertices of $G_1 \dot{\vee} G_2$, its distance matrix D can be written as

$$D = \begin{bmatrix} 2(J-I) & 3J-2R & J \\ 3J-2R^T & 4(J-I)-2B & 2J \\ J & 2J & 2(J-I)-A_2 \end{bmatrix}$$

where J and I denote the all-one matrix and identity matrix, respectively, of appropriate orders. Thus, D is a square matrix of order $p_1 + q_1 + p_2$.

Let $\lambda \neq r_1$ be an eigenvalue of A_1 with an eigenvector X . Then, by the theorem of Perron-Frobenius, X is orthogonal to the all-one matrix J and $A_1X = \lambda X$. Now, by Lemma 1.1, we have

$$\begin{aligned} RR^T &= A_1 + r_1I \\ RR^T X &= (A_1 + r_1I)X \\ &= (\lambda + r_1)X \\ B &= R^T R - 2I \\ BR^T X &= (R^T R - 2I)R^T X \\ &= R^T (A_1 + r_1I)X - 2R^T X \\ &= (\lambda + r_1 - 2)R^T X. \end{aligned}$$

Therefore, $R^T X$ is an eigenvector of B with an eigenvalue $\lambda + r_1 - 2$, which is different from its regularity as $\lambda \neq r_1$. Then, again by the theorem of Perron-Frobenius, $R^T X$ is orthogonal to the all-one vector. Now,

$$\varphi = \begin{bmatrix} X \\ R^T X \\ 0 \end{bmatrix}$$

is an eigenvector of D with an eigenvalue $-2(\lambda + r_1 + 1)$. This is because

$$\begin{aligned}
D\varphi &= \begin{bmatrix} 2(J-I) & 3J-2R & J \\ 3J-2R^T & 4(J-I)-2B & 2J \\ J & 2J & 2(J-I)-A_2 \end{bmatrix} \begin{bmatrix} X \\ R^T X \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} -2X - 2(\lambda + r_1)X \\ -2R^T X - 4R^T X - 2(\lambda + r_1 - 2)R^T X \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} -2(\lambda + r_1 + 1)X \\ -2(\lambda + r_1 + 1)R^T X \\ 0 \end{bmatrix} \\
&= -2(\lambda + r_1 + 1)\varphi.
\end{aligned}$$

Now, let Y be an eigenvector of $A(L(G_1))$ corresponding to the eigenvalue $\lambda + r_1 - 2$, different from the regularity of $L(G_1)$. Then, Y is orthogonal to J . Now, using Lemma 1.1, we can show that

$$\phi = \begin{bmatrix} RY \\ -Y \\ 0 \end{bmatrix}$$

is an eigenvector of D with an eigenvalue 0. This is because

$$\begin{aligned}
D\phi &= \begin{bmatrix} 2(J-I) & 3J-2R & J \\ 3J-2R^T & 4(J-I)-A(L(G_1)) & 2J \\ J & 2J & 2(J-I)-A(G_2) \end{bmatrix} \begin{bmatrix} RY \\ -Y \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} -2RY + 2RY \\ -2R^T RY + 4Y + 2(R^T R - 2I)Y \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
&= 0\phi
\end{aligned}$$

Now, -2 is an eigenvalue of $A(L(G_1))$ with multiplicity $q_1 - p_1$ times. Let Z be an eigenvector of $A(L(G_1))$ with eigenvalue -2 . Then, by Lemma 1.2, $RZ = 0$. Now,

$$\chi = \begin{bmatrix} 0 \\ Z \\ 0 \end{bmatrix}$$

is an eigenvector of D with an eigenvalue 0. This is because

$$\begin{aligned}
D\chi &= \begin{bmatrix} 2(J-I) & 3J-2R & J \\ 3J-2R^T & 4(J-I)-A(L(G_1)) & 2J \\ J & 2J & 2(J-I)-A(G_2) \end{bmatrix} \begin{bmatrix} 0 \\ Z \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
&= 0\chi
\end{aligned}$$

Now, let $\mu \neq r_2$ be an eigenvalue of G_2 with an eigenvector W . Then,

$$\eta = \begin{bmatrix} 0 \\ 0 \\ W \end{bmatrix}$$

is an eigenvector of D with an eigenvalue $-(\mu + 2)$. This is because

$$\begin{aligned}
D\eta &= \begin{bmatrix} 2(J-I) & 3J-2R & J \\ 3J-2R^T & 4(J-I)-A(L(G_1)) & 2J \\ J & 2J & 2(J-I)-A(G_2) \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ W \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ 0 \\ (-2 - \mu)W \end{bmatrix} = -(\mu + 2)\eta
\end{aligned}$$

Thus, we have obtained $p_1 - 1 + p_1 - 1 + q_1 - p_1 + p_2 - 1$, that is, $p_1 + q_1 + p_2 - 3$ eigenvalues and now we will determine the remaining three eigenvalues. We note that all the eigenvectors constructed so far, are orthogonal to

$$\begin{bmatrix} J \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ J \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 0 \\ J \end{bmatrix}.$$

Also, since D is symmetric, $R^{p_1+p_2+q_1}$ has an orthogonal basis consisting of eigenvectors of D and hence the remaining three eigenvectors are spanned by these three vectors and is of the form

$$\rho = \begin{bmatrix} \alpha J \\ \beta J \\ \gamma J \end{bmatrix}$$

for some $(\alpha, \beta, \gamma) \neq (0, 0, 0)$. Thus, if σ is an eigenvalue of D with an eigenvector ρ then from $D\rho = \sigma\rho$, we can see that the remaining three are the eigenvalues of the matrix

$$\begin{bmatrix} 2(p_1 - 1) & 3q_1 - 2r_1 & p_2 \\ 3p_1 - 4 & 4(q_1 - r_1) & 2p_2 \\ p_1 & 2q_1 & 2p_2 - 2 - r_2 \end{bmatrix}$$

whose characteristic equation is

$$x^3 - (2p_1 + 2p_2 + 4q_1 - 4r_1 - r_2 - 4)x^2 + (4p_1 + 4p_2 - 3p_1p_2 + 4q_1 + p_1q_1 - 4p_2q_1 - 8r_1 + 2p_1r_1 + 8p_2r_1 - 2r_2 + 2p_1r_2 + 4q_1r_2 - 4r_1r_2 - 4)x + 2p_1r_1r_2 + p_1q_1r_2 - 4q_1r_2 - 4p_1p_2r_1 + 4p_1r_1 - 2p_1p_2q_1 + 8p_2q_1 + 2p_1q_1 - 8q_1 = 0.$$

This completes the proof. \square

3. The distance spectrum of $G_1 \underline{\vee} G_2$

In this section, we obtain the distance spectrum of $G_1 \underline{\vee} G_2$ for two regular graphs G_1 and G_2 .

Theorem 3.1. *Let G_i be an r_i -regular graph on p_i vertices and q_i edges with the adjacency matrix A_i and adjacency spectrum $\{r_i, \lambda_{i2}, \lambda_{i3}, \dots, \lambda_{ip_i}\}$, $i = 1, 2$. Then, the distance spectrum of $G_1 \underline{\vee} G_2$ consists of the following numbers:*

$$-\left(\lambda_{1j} + 3 \pm \sqrt{(\lambda_{1j} + 1)^2 + 4(\lambda_{1j} + r_1)}\right), j = 2, 3, \dots, p_1; -2 \text{ of multiplicity } q_1 - p_1; -(\lambda_{2j} + 2); j = 2, 3, \dots, p_2,$$

together with the three eigenvalues of

$$\begin{bmatrix} 4(p_1 - 1) - 2r_1 & 3q_1 - 2r_1 & 2p_2 \\ 3p_1 - 4 & 2(q_1 - 1) & p_2 \\ 2p_1 & q_1 & 2(p_2 - 1) - r_2 \end{bmatrix}.$$

Proof. Given that G_1 and G_2 are regular graphs with regularity r_1 and r_2 respectively. Let R be the incidence matrix of G_1 . Then, by a proper ordering of the vertices of $G_1 \underline{\vee} G_2$, its distance matrix D can be written as

$$\begin{aligned} D &= \begin{bmatrix} 2A_1 + 4\bar{A}_1 & 3J - 2R & 2J \\ 3J - 2R^T & 2(J - I) & J \\ 2J & J & A_2 + 2\bar{A}_2 \end{bmatrix} \\ &= \begin{bmatrix} 4(J - I) - 2A_1 & 3J - 2R & 2J \\ 3J - 2R^T & 2(J - I) & J \\ 2J & J & 2(J - I) - A_2 \end{bmatrix}. \end{aligned}$$

Let $\lambda \neq r$ be an eigenvalue of A_1 with an eigenvector X . Then, by Perron-Frobenius theorem X is orthogonal to the all-one matrix J and $A_1X = \lambda X$. Now, we investigate the condition under which

$$\phi = \begin{bmatrix} tX \\ R^T X \\ 0 \end{bmatrix}$$

is an eigenvector of D . If μ is an eigenvalue of D with ϕ as eigenvector, then from the equation $D\phi = \mu\phi$, we get

$$(-4 - 2\lambda)t - 2(\lambda + r_1) = \mu t$$

$$-2t - 2 = \mu$$

$$t^2 - t(\lambda + 1) - (\lambda + r_1) = 0$$

so that t has two values

$$\begin{aligned} t_1 &= \frac{\lambda + 1 + \sqrt{(\lambda + 1)^2 + 4(\lambda + r_1)}}{2}, \\ t_2 &= \frac{\lambda + 1 - \sqrt{(\lambda + 1)^2 + 4(\lambda + r_1)}}{2}. \end{aligned}$$

Thus, corresponding to each eigenvalue $\lambda \neq r_1$ of G_1 , we get two eigenvalues $-2(t_1 + 1)$ and $-2(t_2 + 1)$ and get $2p_1 - 2$ eigenvalues in total.

Now, let Z be an eigenvector of $L(G_1)$ with the eigenvalue -2 . Then, by Lemma 1.2, $RZ = 0$. Now,

$$\psi = \begin{bmatrix} 0 \\ Z \\ 0 \end{bmatrix}$$

is an eigenvector of D with an eigenvalue -2 . This is because

$$D\psi = \begin{bmatrix} 4(J - I) - 2A_1 & 3J - 2R & 2J \\ 3J - 2R^T & 2(J - I) & J \\ 2J & J & 2(J - I) - A_2 \end{bmatrix} \begin{bmatrix} 0 \\ Z \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -2Z \\ 0 \end{bmatrix} = -2\psi$$

Now, let μ be an eigenvalue of G_2 , other than its regularity, with an eigenvector w . Then, it is easy to see that

$$\chi = \begin{bmatrix} 0 \\ 0 \\ w \end{bmatrix}$$

is an eigenvector of D with an eigenvalue $-(\mu + 2)$. Thus, we get $2(p_1 - 1) + q_1 - p_1 + p_2 - 1$, that is, $p_1 + p_2 + q_1 - 3$ eigenvalues of D . Clearly, all the corresponding eigenvectors are orthogonal to

$$\begin{bmatrix} J \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ J \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 0 \\ J \end{bmatrix}.$$

Also, since D is symmetric, $R^{p_1+p_2+q_1}$ has an orthogonal basis consisting of eigenvectors of D . So, the remaining three eigenvectors are spanned by these three vectors and is of the form

$$\nu = \begin{bmatrix} \alpha J \\ \beta J \\ \gamma J \end{bmatrix}$$

for some $(\alpha, \beta, \gamma) \neq (0, 0, 0)$. Therefore, if ν is an eigenvalue of D with an eigenvector ν then from $D\nu = \delta\nu$ we can see that the remaining three are the eigenvalues of the matrix

$$\begin{bmatrix} 4(p_1 - 1) - 2r_1 & 3q_1 - 2r_1 & 2p_2 \\ 3p_1 - 4 & 2(q_1 - 1) & p_2 \\ 2p_1 & q_1 & 2(p_2 - 1) - r_2 \end{bmatrix}.$$

This completes the proof. □

Acknowledgment

The authors are indebted to the anonymous referees for their valuable comments and suggestions which led to an improved presentation of the results.

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