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The hyper-Zagreb index and some Hamiltonian properties of graphs

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Abstract

Let $G = (V(G), E(G))$ be a finite undirected graph without loops or multiple edges. The hyper-Zagreb index of G, denoted $HZ(G)$, is defined as $\sum_{uv \in E(G)} (d_G(u) + d_G(v))^2$, where $d_G(u)$ denotes the degree of a vertex $u \in V(G)$. Using the hyper-Zagreb index of the complement of a graph, several sufficient conditions for some Hamiltonian properties of the graph are presented in this paper.

Keywords: topological index; hyper-Zagreb index; Hamiltonian property.

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1. Introduction and statements of the results

In this paper, only finite undirected graphs without loops or multiple edges are considered. Notation and terminology not defined here follow that described in [2]. Let $G = (V(G), E(G))$ be a graph. Denote by n, m, δ, and κ the order, size, minimum degree, and connectivity of G, respectively. The complement of G is denoted by G^c. The hyper-Zagreb index of G, denoted $HZ(G)$, is defined as $\sum_{uv \in E(G)} (d_G(u) + d_G(v))^2$ (see [11]). It needs to be mentioned here that the hyper-Zagreb index of G is actually equal to $F(G) + 2M_2(G)$, where $F(G)$ is the forgotten topological index of G (see [5]) and $M_2(G)$ is the second Zagreb index of G (see [9]). Denote by $\mu_n(G)$ the largest eigenvalue of the adjacency matrix of a graph G of order n. For two disjoint graphs G_1 and G_2, the union and join of G_1 and G_2 are denoted by $G_1 + G_2$ and $G_1 \vee G_2$, respectively. Denote by sK_1 the union of s isolated vertices. The concept of closure of a graph G was introduced by Bondy and Chvátal in [1]. The k-closure of a graph G, denoted $cl_k(G)$, is a graph obtained from G by recursively joining two nonadjacent vertices such that their degree sum is at least k until no such pair remains. Denote by $C(n, r)$ the number of r-combinations of a set with n distinct elements.

A cycle C in a graph G is called a Hamiltonian cycle of G if C contains all the vertices of G. A graph G is called Hamiltonian if G has a Hamiltonian cycle. A path P in a graph G is called a Hamiltonian path of G if P contains all the vertices of G. A graph G is called traceable if G has a Hamiltonian path. The Hamiltonicity and traceability of graphs have been heavily investigated in last several decades. Lot of results in this direction can be found in the survey papers [6–8].

In 2010, Fiedler and Nikiforov obtained the following spectral conditions for the Hamiltonicity and traceability of graphs.

Theorem 1.1. [4] *Let G be a graph of order n.*

(a) *If $\mu_n(G^c) \le \sqrt{n-1}$, then G contains a Hamiltonian path unless $G \cong K_{n-1} + K_1$.*

(b) *If $\mu_n(G^c) \le \sqrt{n-2}$, then G contains a Hamiltonian cycle unless $G \cong K_1 \vee (K_1 + K_{n-2})$.*

Using the ideas and techniques developed by Fiedler and Nikiforov in [4], we in this note present the following sufficient conditions for the Hamiltonian and traceable graphs. Those conditions involve the hyper-Zagreb index of the complements of the graphs.

Theorem 1.2. *Let G be a graph of order n.*

(a) *If $n \ge 3$ and $HZ(G^c) \le (n-1)^2(n-2)$, then G is Hamiltonian, $G \cong K_1 \vee (K_1 + K_{n-2})$, or $G \cong K_2 \vee (3K_1)$.*

(b) *If G is 2-connected with $n \ge 12$ and $HZ(G^c) \le (n-1)^2(2n-7)$, then G is Hamiltonian.*

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(c) *If G is 3-connected with $n \geq 18$ and $HZ(G^c) \leq (n-1)^2(3n-15)$, then G is Hamiltonian.*

(d) *If $n \geq 3$, $\kappa \geq 2$, and $2HZ(G^c) \leq (n-1)^2(n(n-1)-(n-\kappa-1)(n+\kappa))$, then G is Hamiltonian or $G \cong K_\kappa \vee ((\kappa+1)K_1)$.*

Theorem 1.3. *Let G be a graph of order n.*

(a) *If $n \geq 4$, $\delta \geq 1$, and $HZ(G^c) \leq n^2(2n-5)$, then G is traceable or G belongs to the class $\{K_1 \vee (3K_1), K_2 \vee (4K_1), K_3 \vee (5K_1), K_4 \vee (6K_1)\}$.*

(b) *If $n \geq 2$, $\kappa \geq 1$, and $2HZ(G^c) \leq n^2(n(n-1)-(n-\kappa-2)(n+\kappa+1))$, then G is traceable or $G \cong K_\kappa \vee ((\kappa+2)K_1)$.*

2. Previous results

We need the following previous results as lemmas to prove our theorems.

Lemma 2.1. (Lemma 2.1 on Page 112 in [1]) *A graph G of order n is Hamiltonian if and only if $cl_n(G)$ is Hamiltonian.*

Lemma 2.2. (Lemma 2.4 on Page 113 in [1]) *A graph G of order n is traceable if and only if $cl_{n-1}(G)$ is traceable.*

Lemma 2.3. (Corollary 4.6 on Page 60 in [2]) *Let G be a graph of order $n \geq 3$. If $m \geq C(n-1,2)+1$, then G is Hamiltonian, $G \cong K_1 \vee (K_1 + K_{n-2})$, or $G \cong K_2 \vee (3K_1)$.*

Lemma 2.4. (Theorem 11 on Page 1575 in [3]) *Let G be a 2-connected graph of order $n \geq 12$. If $m \geq C(n-2,2)+4$, then G is Hamiltonian or $G \cong K_2 \vee ((2K_1)+K_{n-4})$.*

Lemma 2.5. (Theorem 10 on Page 1574 in [3]) *let G be a 3-connected graph of order $n \geq 18$. If $m \geq C(n-3,2)+9$, then G is Hamiltonian or $G \cong K_3 \vee ((3K_1)+K_{n-6})$.*

Lemma 2.6. (Theorem 1 in [10]) *Let G be a graph of order $n \geq 3$, m edges, and connectivity $\kappa \geq 2$. If $m \geq (n-\kappa-1)(n+\kappa)/2$, then G is Hamiltonian or $G \cong K_\kappa \vee ((\kappa+1)K_1)$.*

Lemma 2.7. (Lemma 4 in [12]) *Let G be a graph of order $n \geq 4$, m edges, and $\delta \geq 1$. If $m \geq C(n-2,2)+2$, then G is traceable or G belongs to the class $\{K_1 \vee ((2K_1)+K_{n-3}), K_1 \vee (K_1 + K_{1,3}), K_{2,4}, K_2 \vee (4K_1), (K_2 \vee ((3K_1)+K_2), K_1 \vee K_{2,5}, K_3 \vee (5K_1), (K_2 \vee (K_1 + K_{1,4}), K_4 \vee (6K_1)\}$.*

Lemma 2.8. (Theorem 2 in [10]) *Let G be a graph of order $n \geq 2$, m edges, and connectivity $\kappa \geq 1$. If $m \geq (n-\kappa-2)(n+\kappa+1)/2$, then G is traceable or $G \cong K_\kappa \vee ((\kappa+2)K_1)$.*

3. Proofs

Proof of Theorem 1.2. Let G be a graph satisfying the conditions of part (a), part (b), part (c), or part (d) of Theorem 1.2 and G is not Hamiltonian. Then Lemma 2.1 implies that $H := cl_n(G)$ is not Hamiltonian and therefore H is not K_n. Thus there exist two vertices x and y in $V(H)$ such that $xy \notin E(H)$ and for any pair of nonadjacent vertices u and v in $V(H)$ we have $d_H(u) + d_H(v) \leq n-1$. Hence, for any pair of adjacent vertices u and v in $V(H^c)$, it holds that

$$d_{H^c}(u) + d_{H^c}(v) = n-1-d_H(u)+n-1-d_H(v) \geq n-1.$$

Thus,

$$(d_{H^c}(u) + d_{H^c}(v))^2 \geq (n-1)^2.$$

Therefore,

$$HZ(H^c) = \sum_{uv \in E(H^c)} (d_{H^c}(u) + d_{H^c}(v))^2 \geq (n-1)^2 m(H^c).$$

Notice that $d_{H^c}(u) \leq d_{G^c}(u)$ for each vertex $u \in V(H^c) = V(G^c)$ and $E(H^c) \subseteq E(G^c)$. We have that

$$HZ(G^c) \geq HZ(H^c) \geq (n-1)^2 m(H^c).$$

(a) Suppose G satisfies conditions of Theorem 1.2(a). Notice that H is not Hamiltonian. Lemma 2.3 implies that

$$m(H) \leq C(n-1,2),$$

or

$$m(H) = C(n-1,2)+1 \quad \text{and} \quad H \cong K_1 \vee (K_1 + K_{n-2}),$$

or

$$m(H) = C(n-1,2)+1 \quad \text{and} \quad H \cong K_2 \vee (3K_1).$$

If $m(H) \le C(n-1,2)$, then

$$(n-1)^2(n-2) \ge HZ(G^c) \ge HZ(H^c) \ge (n-1)^2 m(H^c) \ge (n-1)^2 \left(C(n,2) - C(n-1,2)\right) = (n-1)^3,$$

which is a contradiction.
If $H \cong K_1 \vee (K_1 + K_{n-2})$ or $H \cong K_2 \vee (3K_1)$, then

$$(n-1)^2(n-2) \ge HZ(G^c) \ge HZ(H^c) \ge (n-1)^2 m(H^c) \ge (n-1)^2 \left(C(n,2) - C(n-1,2) - 1\right) = (n-1)^2(n-2).$$

Thus $d_{G^c}(u) = d_{H^c}(u)$ for each $u \in V(H^c) = V(G^c)$, $d_{G^c}(u)d_{G^c}(v) = d_{H^c}(u)d_{H^c}(v)$ for each $uv \in E(H^c)$, and $d_{H^c}(u) + d_{H^c}(v) = n-1$ for each $uv \in E(H^c)$. Therefore $G = H$.

(b) Suppose G satisfies conditions of Theorem 1.2(b). Notice that H is not Hamiltonian. Lemma 2.4 implies that

$$m(H) \le C(n-2,2)+3$$

or

$$m(H) = C(n-2,2)+4 \quad \text{and} \quad H \cong K_2 \vee (2K_1 + K_{n-4}).$$

If $m(H) \le C(n-2,2)+3$, then

$$(n-1)^2(2n-7) \ge HZ(G^c) \ge HZ(H^c) \ge (n-1)^2 m(H^c) \ge (n-1)^2 \left(C(n,2) - C(n-2,2) - 3\right) = (n-1)^2(2n-6),$$

that is a contradiction.
If $H \cong K_2 \vee (2K_1 + K_{n-4})$, then

$$(n-1)^2(2n-7) \ge HZ(G^c) \ge HZ(H^c) \ge (n-1)^2 m(H^c) \ge (n-1)^2 \left(C(n,2) - C(n-2,2) - 4\right) = (n-1)^2(2n-7).$$

Thus $d_{G^c}(u) = d_{H^c}(u)$ for each $u \in V(H^c) = V(G^c)$, $d_{G^c}(u)d_{G^c}(v) = d_{H^c}(u)d_{H^c}(v)$ for each $uv \in E(H^c)$, and $d_{H^c}(u) + d_{H^c}(v) = n-1$ for each $uv \in E(H^c)$. Therefore, $G = H \cong K_2 \vee (3K_1)$. Since $n \ge 12$, a contradiction is obtained.

(c) Suppose G satisfies conditions of Theorem 1.2(c). Notice that H is not Hamiltonian. Lemma 2.5 implies that $m(H) \le C(n-3,2)+8$ or $m(H) = C(n-3,2)+9$ and $H \cong K_3 \vee (3K_1 + K_{n-6})$.
If $m(H) \le C(n-3,2)+8$, then

$$(n-1)^2(3n-15) \ge HZ(G^c) \ge HZ(H^c) \ge (n-1)^2 m(H^c) \ge (n-1)^2 \left(C(n,2) - C(n-3,2) - 8\right) = (n-1)^2(3n-14),$$

a contradiction.
If $H \cong K_3 \vee (3K_1 + K_{n-6}))$, then

$$(n-1)^2(3n-15) \ge HZ(G^c) \ge HZ(H^c) \ge (n-1)^2 m(H^c) \ge (n-1)^2 \left(C(n,2) - C(n-3,2) - 9\right) = (n-1)^2(3n-15).$$

Thus $d_{G^c}(u) = d_{H^c}(u)$ for each $u \in V(H^c) = V(G^c)$, $d_{G^c}(u)d_{G^c}(v) = d_{H^c}(u)d_{H^c}(v)$ for each $uv \in E(H^c)$, and $d_{H^c}(u) + d_{H^c}(v) = n-1$ for each $uv \in E(H^c)$. Therefore $G = H \cong K_3 \vee (4K_1)$. Since $n \ge 18$, we reach at a contradiction.

(d) Suppose G satisfies conditions of Theorem 1.2(d). Note that H is not Hamiltonian. Lemma 2.6 implies that

$$2m(H) \le (n-\kappa-1)(n+\kappa) - 1$$

or

$$2m(H) = (n-\kappa-1)(n+\kappa) \quad \text{and} \quad H \cong K_\kappa \vee ((\kappa+1)K_1).$$

If $2m(H) \le (n-\kappa-1)(n+\kappa) - 1$, then

$$\begin{aligned}(n-1)^2(n(n-1) - (n-\kappa-1)(n+\kappa)) &\ge 2HZ(G^c) \ge 2HZ(H^c) \ge 2(n-1)^2 m(H^c) \\ &\ge (n-1)^2 \left(2C(n,2) - (n-\kappa-1)(n+\kappa) + 1\right) \\ &= (n-1)^2(n(n-1) - (n-\kappa-1)(n+\kappa) + 1),\end{aligned}$$

which is a contradiction.

If $H \cong K_\kappa \vee ((\kappa+1)K_1)$, then

$$(n-1)^2(n(n-1)-(n-\kappa-1)(n+\kappa)) \geq 2HZ(G^c) \geq 2HZ(H^c) \geq 2(n-1)^2 m(H^c)$$
$$\geq (n-1)^2 (2C(n,2)-(n-\kappa-1)(n+\kappa))$$
$$= (n-1)^2(n(n-1)-(n-\kappa-1)(n+\kappa)).$$

Thus $d_{G^c}(u) = d_{H^c}(u)$ for each $u \in V(H^c) = V(G^c)$, $d_{G^c}(u)d_{G^c}(v) = d_{H^c}(u)d_{H^c}(v)$ for each $uv \in E(H^c)$, and $d_{H^c}(u) + d_{H^c}(v) = n-1$ for each $uv \in E(H^c)$. Therefore $G = H$.

Proof of Theorem 1.3. Let G be a graph satisfying the conditions of part (a) or part (b) of Theorem 1.3 and G is not traceable. Then Lemma 2.2 implies that $H := cl_{n-1}(G)$ is not traceable and therefore H is not K_n. Thus there exist two vertices x and y in $V(H)$ such that $xy \notin E(H)$ and for any pair of nonadjacent vertices u and v in $V(H)$ we have

$$d_H(u) + d_H(v) \leq n-2.$$

Hence, for any pair of adjacent vertices u and v in $V(H^c)$, we have that

$$d_{H^c}(u) + d_{H^c}(v) = n-1-d_H(u)+n-1-d_H(v) \geq n.$$

Thus,

$$(d_{H^c}(u) + d_{H^c}(v))^2 \geq n^2.$$

Therefore,

$$HZ(H^c) = \sum_{uv \in E(H^c)} (d_{H^c}(u) + d_{H^c}(v))^2 \geq n^2 m(H^c).$$

Notice that $d_{H^c}(u) \leq d_{G^c}(u)$ for each $u \in V(H^c) = V(G^c)$ and $E(H^c) \subseteq E(G^c)$. We have that

$$HZ(G^c) \geq HZ(H^c) \geq n^2 m(H^c).$$

(a) Suppose G satisfies conditions of Theorem 1.3(a). Notice that H is not traceable. Lemma 2.7 implies that $m(H) \leq C(n-2,2)+1$ or $m(H) = C(n-2,2)+2$ and H belongs to the class $\{K_1 \vee ((2K_1)+K_{n-3}), K_1 \vee (K_1 + K_{1,3}), K_{2,4}, K_2 \vee (4K_1), (K_2 \vee ((3K_1)+K_2), K_1 \vee K_{2,5}, K_3 \vee (5K_1), (K_2 \vee (K_1 + K_{1,4}), K_4 \vee (6K_1)\}$.
If $m(H) \leq C(n-2,2)+1$, then it holds that

$$n^2(2n-5) \geq HZ(G^c) \geq HZ(H^c) \geq n^2 m(H^c) \geq n^2 (C(n,2) - C(n-2,2) - 1) = n^2(2n-4),$$

which is a contradiction.
If H belongs to the class $\{K_1 \vee ((2K_1)+K_{n-3}), K_1 \vee (K_1 + K_{1,3}), K_{2,4}, K_2 \vee (4K_1), (K_2 \vee ((3K_1)+K_2), K_1 \vee K_{2,5}, K_3 \vee (5K_1), (K_2 \vee (K_1 + K_{1,4}), K_4 \vee (6K_1)\}$, then

$$n^2(2n-5) \geq HZ(G^c) \geq HZ(H^c) \geq n^2 m(H^c) \geq n^2 (C(n,2) - C(n-2,2) - 2) = n^2(2n-5).$$

Thus $d_{G^c}(u) = d_{H^c}(u)$ for each $u \in V(H^c) = V(G^c)$, $d_{G^c}(u)d_{G^c}(v) = d_{H^c}(u)d_{H^c}(v)$ for each $uv \in E(H^c)$, and $d_{H^c}(u) + d_{H^c}(v) = n$ for each $uv \in E(H^c)$. Therefore $G = H$ and H belongs to the class $\{K_1 \vee (3K_1), K_2 \vee (4K_1), K_3 \vee (5K_1), K_4 \vee (6K_1)\}$.

(b) Suppose G satisfies conditions of Theorem 1.3(b). Notice that H is not traceable. Lemma 2.8 implies that

$$2m(H) \leq (n-\kappa-2)(n+\kappa+1)-1$$

or

$$2m(H) = (n-\kappa-2)(n+\kappa+1) \quad \text{and} \quad H \cong K_\kappa \vee ((\kappa+2)K_1).$$

If $2m(H) \leq (n-\kappa-2)(n+\kappa+1)-1$, then

$$n^2(n(n-1)-(n-\kappa-2)(n+\kappa+1)) \geq 2HZ(G^c) \geq 2HZ(H^c) \geq 2n^2 m(H^c) \geq n^2 (2C(n,2)-(n-\kappa-2)(n+\kappa+1)+1)$$
$$= n^2(n(n-1)-(n-\kappa-2)(n+\kappa+1)+1),$$

a contradiction.
If $H \cong K_\kappa \vee ((\kappa+2)K_1)$, then

$$n^2(n(n-1)-(n-\kappa-2)(n+\kappa+1)) \geq 2HZ(G^c) \geq 2HZ(H^c) \geq 2n^2 m(H^c) \geq n^2 (2C(n,2)-(n-\kappa-2)(n+\kappa+1))$$
$$= n^2(n(n-1)-(n-\kappa-2)(n+\kappa+1)).$$

Thus, it holds that $d_{G^c}(u) = d_{H^c}(u)$ for each vertex $u \in V(H^c) = V(G^c)$, $d_{G^c}(u)d_{G^c}(v) = d_{H^c}(u)d_{H^c}(v)$ for each edge $uv \in E(H^c)$, and $d_{H^c}(u) + d_{H^c}(v) = n$ for each edge $uv \in E(H^c)$. Therefore, $G = H$.

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