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On the beta-number and gamma-number of galaxies

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Abstract

The beta-number of a graph G is the smallest positive integer n for which there exists an injective function $f : V(G) \to \{0, 1, \ldots, n\}$ such that each $uv \in E(G)$ is labeled $|f(u) - f(v)|$ and the resulting set of edge labels is $\{c, c+1, \ldots, c+|E(G)|-1\}$ for some positive integer c. The beta-number of G is $+\infty$, otherwise. If $c = 1$, then the resulting beta-number is called the strong beta-number of G. The gamma-number $\gamma(G)$ of a graph G is the smallest positive integer n for which there exists an injective function $f : V(G) \to [0, n]$ such that each $uv \in E(G)$ is labeled $|f(u) - f(v)|$ and the resulting edge labels are distinct. A galaxy is a forest for which each component is a star. In this paper, we determine formulas for the (strong) beta-number and gamma-number of galaxies with five components. As a corollary of these results, we provide formulas for the beta-number and gamma-number of the disjoint union of multiple copies of the same galaxies if the number of copies is odd. Based on this work, we propose a new conjecture on the beta-number of galaxies.

Keywords: beta-number; strong beta-number; gamma-number; β-valuation; graceful labeling.

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1. Introduction

All graphs considered in this paper are finite and undirected without loops or multiple edges. The *vertex set* of a graph G is denoted by $V(G)$, while the *edge set* is denoted by $E(G)$. The *union* $G_1 \cup G_2$ of two subgraphs G_1 and G_2 of a graph G is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. The union of any finite number of subgraphs is defined similarly.

For integers a and b with $a \leq b$, we will denote the set $\{x \in \mathbb{Z} : a \leq x \leq b\}$ by writing $[a, b]$, where $\mathbb{Z}$ denotes the set of all integers. On the other hand, if $a > b$, then we treat $[a, b]$ as the empty set. If such situation appears in particular formulas for a given vertex labeling, then we ignore the corresponding portions of the formulas.

As a means of attacking graph decomposition problems, β-valuations were originated by Rosa [11]. For a graph G of size q, an injective function $f : V(G) \to [0, q]$ is called a β-*valuation* if each $uv \in E(G)$ is labeled $|f(u) - f(v)|$ and the resulting edge labels are distinct. Such a valuation is now commonly known as a *graceful labeling* (the term was coined by Golomb [3]) and a graph with a graceful labeling is called *graceful*. Graceful labelings have been a major focus of attention for many papers. For recent contributions to this subject and other types of labelings, the authors refer the reader to the survey by Gallian [2].

The *gamma-number* $\gamma(G)$ of a graph G with $V(G) = \{v_i : i \in [1, p]\}$ is the smallest positive integer n for which there exists an injective function $f : V(G) \to [0, n]$ such that each $uv \in E(G)$ is labeled $|f(u) - f(v)|$ and the resulting edge labels are distinct. Such functions always exist, one of which is to label v_i by $2^{i-1} - 1$. Hence, every graph G of order p, $\gamma(G) \leq 2^{p-1} - 1$, which shows that $\gamma(G) < +\infty$. If G is a graph of size q with $\gamma(G) = q$, then G is graceful. Thus, the gamma-number of a graph G is a measure of how close G is to being graceful. By definition, it is possible to label the vertices of a graph G with distinct elements of the set $[0, \gamma(G)]$ so that the edges of G receive distinct labels. Of course, some vertex of G must be labeled $\gamma(G)$; however, it is not known whether an edge of G must then be labeled $\gamma(G)$. This concept was introduced by Golomb [3]. At that time and in succeeding years, this concept has been studied and referred to using different terminology. In fact, the gamma-number $\gamma(G)$ of a graph G has often been called the *gracefulness* of G.

A number of authors have invented analogues of gamma-number. For instance, the beta-number and strong beta-number introduced in [10] are such type of parameters. The *beta-number* $\beta(G)$ of a graph G with q edges is the

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smallest positive integer n for which there exists an injective function $f : V(G) \to [0, n]$ such that each $uv \in E(G)$ is labeled $|f(u) - f(v)|$ and the resulting set of edge labels is $[c, c+q-1]$ for some positive integer c. The beta-number of G is $+\infty$ if there exists no such integer n. If $c = 1$, then the resulting beta-number is called the *strong beta-number* of G and is denoted by $\beta_s(G)$.

The following lemma taken from [10] describes how the parameters discussed so far are related.

Lemma 1.1. *For every graph G of order p and size q,*

$$\max\{p-1, q\} \leq \gamma(G) \leq \beta(G) \leq \beta_s(G).$$

At this point, it is convenient to introduce some additional concepts and notation. The *star* with $n+1$ vertices is denoted by S_n. A *galaxy* is a forest for which each component is a star. For the sake of notational convenience, we will denote the galaxy $S_{n_1} \cup S_{n_2} \cup \cdots \cup S_{n_k}$ by simply writing $S(n_1, n_2, \ldots, n_k)$, where $k \geq 2$.

In [10], it was determined the exact values of the (strong) beta-number of several classes of graphs including galaxies with two components, and proved that every nontrivial tree and forest has finite strong beta-number. In [4] the (strong) beta-number for forests with isomorphic components was studied and conjectured that the (strong) beta-number and gamma-number of a forest of order p are either $p-1$ or p. In [5] formulas for the (strong) beta-number of galaxies with three and four components were found, and the following lower bound for the strong beta-number of galaxies was established.

Theorem 1.1. *Let $G \cong S(n_1, n_2, \ldots, n_k)$, where $n_1 n_2 \cdots n_k$ is odd, and $k \equiv 2$ or $3 \pmod 4$. Then*

$$\beta_s(G) \geq \sigma_k + k,$$

where $\sigma_k = n_1 + n_2 + \cdots + n_k$.

In this paper, we determine formulas for the (strong) beta-number and gamma-number of galaxies with five components. As a corollary of these results, we provide formulas for the beta-number and gamma-number of the disjoint union of multiple copies of the same galaxies if the number of copies is odd. These results add credence to the mentioned conjectures, and lead us to propose a new conjecture on the beta-number of galaxies.

There are other kinds of parameters that measure how close a graph is to being graceful. For further knowledge on the (strong) beta-number of graphs and related concepts, the authors suggest that the reader consults the results contained in [1, 5, 7–9, 12]. For the most recent advances on the mentioned conjectures on forests, the authors also direct the reader to the papers [4, 6, 8].

To end this introduction, we state a result found in [4], which will prove to be useful later.

Theorem 1.2. *If F is a forest of order p such that $\beta(F) = p-1$, then*

$$\beta(mF) = mp - 1$$

when m is odd.

2. Results on galaxies with five components

In this section, we present formulas for the (strong) beta-number of galaxies with five components and related results. We start with the following theorem.

Theorem 2.1. *Let $G \cong S(n_1, n_2, n_3, n_4, n_5)$, where n_1, n_2, n_3, n_4 and n_5 are positive integers with $\sigma_5 = n_1 + n_2 + n_3 + n_4 + n_5$. Then*

$$\beta_s(G) = \sigma_5 + 4.$$

Proof. Define the galaxy G with

$$V(G) = \{x_i : i \in [1,5]\} \cup \left(\bigcup_{i=1}^{5} \left\{ y_i^j : j \in [1, n_i] \right\} \right)$$

and

$$E(G) = \bigcup_{i=1}^{5} \left\{ x_i y_i^j : j \in [1, n_i] \right\}.$$

In light of Lemma 1.1, it suffices to show that $\beta_s(G) \leq \sigma_5 + 4$. Thus, consider the following four cases for the vertex labeling $f : V(G) \to [0, \sigma_5 + 4]$.

Case 1. For $n_1 = 2k_1$, $n_2 = 2k_2$, $n_3 = 2k_3$, $n_4 = 2k_4$ and $n_5 = 2k_5$, where k_i is a positive integer for each $i \in [1, 5]$, let

$$f(x_i) = \begin{cases} i-1 & \text{if } i \in [1,3], \\ 2k_1+2k_2+2k_3+2k_4+2k_5+i-1 & \text{if } i \in [4,5], \end{cases}$$

$$f\left(y_1^j\right) = \begin{cases} k_2+k_3+3 & \text{if } j = 1, \\ k_2+k_3+k_5+j+2 & \text{if } j \in [2,k_1], \\ k_2+k_3+2k_4+k_5+j & \text{if } j \in [k_1+1,2k_1], \end{cases}$$

$$f\left(y_2^j\right) = \begin{cases} k_3+j+2 & \text{if } j \in [1,k_2], \\ 2k_1+k_3+2k_4+2k_5+j+1 & \text{if } j \in [k_2+1,2k_2], \end{cases}$$

$$f\left(y_3^j\right) = \begin{cases} j+2 & \text{if } j \in [1,k_3], \\ 2k_1+2k_2+2k_4+2k_5+j+2 & \text{if } j \in [k_3+1,2k_3], \end{cases}$$

$$f\left(y_4^j\right) = \begin{cases} k_1+k_2+k_3+k_5+j+2 & \text{if } j \in [1,2k_4-2], \\ 2k_1+k_2+k_3+2k_4+2k_5+1 & \text{if } j = 2k_4-1, \\ 2k_1+2k_2+k_3+2k_4+2k_5+2 & \text{if } j = 2k_4, \end{cases}$$

$$f\left(y_5^j\right) = \begin{cases} k_2+k_3+j+3 & \text{if } j \in [1,k_5], \\ 2k_1+k_2+k_3+2k_4+j & \text{if } j \in [k_5+1,2k_5]. \end{cases}$$

Case 2. For $n_1 = 2k_1 - 1$, $n_2 = 2k_2$, $n_3 = 2k_3$, $n_4 = 2k_4$ and $n_5 = k_5$, where k_i is a positive integer for each $i \in [1, 5]$, let

$$f(x_i) = \begin{cases} i-1 & \text{if } i \in [1,3], \\ 2k_1+2k_2+2k_3+2k_4+k_5+i-2 & \text{if } i \in [4,5], \end{cases}$$

$$f\left(y_1^j\right) = \begin{cases} k_2+k_3+k_4+j+1 & \text{if } j \in [1,k_1], \\ k_2+k_3+k_4+k_5+j+1 & \text{if } j \in [k_1+1,2k_1-1], \end{cases}$$

$$f\left(y_2^j\right) = \begin{cases} k_3+j+2 & \text{if } j \in [1,k_2], \\ 2k_1+k_3+2k_4+k_5+j & \text{if } j \in [k_2+1,2k_2], \end{cases}$$

$$f\left(y_3^j\right) = \begin{cases} j+2 & \text{if } j \in [1,k_3], \\ 2k_1+2k_2+2k_4+k_5+j+1 & \text{if } j \in [k_3+1,2k_3], \end{cases}$$

$$f\left(y_4^j\right) = \begin{cases} k_2+k_3+j+2 & \text{if } j \in [1,k_4-1], \\ 2k_1+k_2+k_3+k_5+j+1 & \text{if } j \in [k_4,2k_4-1], \\ 2k_1+2k_2+k_3+2k_4+k_5+1 & \text{if } j = 2k_4, \end{cases}$$

$$f\left(y_5^j\right) = k_1+k_2+k_3+k_4+j+1 \quad \text{if } j \in [1,k_5].$$

Case 3. For $n_1 = 2k_1 - 1$, $n_2 = 2k_2 - 1$, $n_3 = 2k_3$, $n_4 = k_4$ and $n_5 = 2k_5 - 1$, where k_i is a positive integer for each $i \in [1, 5]$, let

$$f(x_i) = \begin{cases} i-1 & \text{if } i \in [1,3], \\ 2k_1+2k_2+2k_3+k_4+2k_5+i-4 & \text{if } i \in [4,5], \end{cases}$$

$$f\left(y_1^j\right) = \begin{cases} k_2+k_3+k_5+j+1 & \text{if } j \in [1,k_1-1], \\ k_2+k_3+k_4+k_5+j & \text{if } j \in [k_1,2k_1-1], \end{cases}$$

$$f\left(y_2^j\right) = \begin{cases} k_3 + j + 2 & \text{if } j \in [1, k_2], \\ 2k_1 + k_3 + k_4 + 2k_5 + j - 1 & \text{if } j \in [k_2 + 1, 2k_2 - 1], \end{cases}$$

$$f\left(y_3^j\right) = \begin{cases} j + 2 & \text{if } j \in [1, k_3], \\ 2k_1 + 2k_2 + k_4 + 2k_5 + j - 1 & \text{if } j \in [k_3 + 1, 2k_3], \end{cases}$$

$$f\left(y_4^j\right) = \begin{cases} k_1 + k_2 + k_3 + k_5 + j & \text{if } j \in [1, k_4 - 1], \\ 2k_1 + 2k_2 + k_3 + k_4 + 2k_5 - 1 & \text{if } j = k_4, \end{cases}$$

$$f\left(y_5^j\right) = \begin{cases} k_2 + k_3 + j + 2 & \text{if } j \in [1, k_5 - 1], \\ 2k_1 + k_2 + k_3 + k_4 + j & \text{if } j \in [k_5, 2k_5 - 1]. \end{cases}$$

Case 4. For $n_1 = 2k_1 - 1$, $n_2 = 2k_2 - 1$, $n_3 = 2k_3 - 1$, $n_4 = 2k_4 - 1$ and $n_5 = 2k_5 - 1$, where k_i is a positive integer for each $i \in [1, 5]$, let

$$f(x_i) = \begin{cases} i - 1 & \text{if } i \in [1, 3], \\ 2k_1 + 2k_2 + 2k_3 + 2k_4 + 2k_5 + i - 6 & \text{if } i \in [4, 5], \end{cases}$$

$$f\left(y_1^j\right) = \begin{cases} k_2 + k_3 + k_4 + j & \text{if } j \in [1, k_1], \\ k_2 + k_3 + k_4 + 2k_5 + j - 2 & \text{if } j \in [k_1 + 1, 2k_1 - 1], \end{cases}$$

$$f\left(y_2^j\right) = \begin{cases} k_3 + j + 2 & \text{if } j \in [1, k_2 - 1], \\ 2k_1 + k_3 + 2k_4 + 2k_5 + j - 2 & \text{if } j \in [k_2, 2k_2 - 1], \end{cases}$$

$$f\left(y_3^j\right) = \begin{cases} j + 2 & \text{if } j \in [1, k_3], \\ 2k_1 + 2k_2 + 2k_4 + 2k_5 + j - 2 & \text{if } j \in [k_3 + 1, 2k_3 - 1], \end{cases}$$

$$f\left(y_4^j\right) = \begin{cases} k_2 + k_3 + j + 1 & \text{if } j \in [1, k_4 - 1], \\ 2k_1 + k_2 + k_3 + 2k_5 + j - 2 & \text{if } j \in [k_4, 2k_4 - 1], \end{cases}$$

$$f\left(y_5^j\right) = \begin{cases} k_1 + k_2 + k_3 + k_4 + j & \text{if } j \in [1, 2k_5 - 2], \\ 2k_1 + 2k_2 + k_3 + 2k_4 + 2k_5 - 2 & \text{if } j = 2k_5 - 1. \end{cases}$$

Therefore, f satisfies the necessary requirements for $\beta_s(G)$, which implies that $\beta_s(G) \le \sigma_5 + 4$. □

As a simple consequence of Lemma 1.1 and Theorem 2.1, we have the following result.

Corollary 2.1. *Let $G \cong S(n_1, n_2, n_3, n_4, n_5)$, where n_1, n_2, n_3, n_4 and n_5 are positive integers with $\sigma_5 = n_1 + n_2 + n_3 + n_4 + n_5$. Then*

$$\beta(G) = \gamma(G) = \sigma_5 + 4.$$

Applying Theorem 1.2 with Corollary 2.1, we obtain the following result.

Corollary 2.2. *Let $G \cong S(n_1, n_2, n_3, n_4, n_5)$, where m is odd and n_1, n_2, n_3, n_4, n_5 are positive integers with $\sigma_5 = n_1 + n_2 + n_3 + n_4 + n_5$. Then*

$$\beta(m(G)) = \gamma(m(G)) = m(\sigma_5 + 5) - 1.$$

3. Conclusion

It was determined in [4] that $\beta_s(S(n_1, n_2, n_3)) = \sigma_3 + 2$ if $n_1 n_2 n_3$ is even, $\beta_s(S(n_1, n_2, n_3)) = \sigma_3 + 2$ if $n_1 n_2 n_3$ is odd and $\beta_s(S(n_1, n_2, n_3, n_4)) = \sigma_4 + 3$ for positive integers n_1, n_2, n_3 and n_4, and proposed the following conjecture.

Conjecture 3.1. *Let $G \cong S(n_1, n_2, \ldots, n_k)$, where n_1, n_2, …, n_k are positive integers with $\sigma_k = n_1 + n_2 + \cdots + n_k$. Then*

$$\beta_s(G) = \begin{cases} \sigma_k + k & \text{if } n_1 n_2 \cdots n_k \text{ is even, or } k \equiv 0 \text{ or } 1 \pmod 4, \\ \sigma_k + k - 1 & \text{if } n_1 n_2 \cdots n_k \text{ is odd, and } k \equiv 2 \text{ or } 3 \pmod 4. \end{cases}$$

To put forth a conjecture for the beta-number of galaxies, we now make some remarks. It is known from [6] that $\beta(S(n_1,n_2)) = \sigma_2 + 1$ if $n_1 n_2$ is even, and $\beta(S(n_1,n_2)) = \sigma_2 + 2$ if $n_1 n_2$ is odd. It is also known from [5] that $\beta(S(n_1,n_2,n_3)) = \sigma_3 + 2$ and $\beta(S(n_1,n_2,n_3,n_4)) = \sigma_4 + 3$ for positive integers n_1, n_2, n_3 and n_4. Therefore, all these facts together with the results in this paper lead us to propose the following conjecture.

Conjecture 3.2. *Let $G \cong S(n_1,n_2,\ldots,n_k)$, where n_1, n_2, …, n_k are positive integers with $\sigma_k = n_1 + n_2 + \cdots + n_k$. Then*

$$\beta(G) = \begin{cases} \sigma_k + k & \text{if } n_1 n_2 \cdots n_k \text{ is odd and } k \equiv 2 \pmod 4, \\ \sigma_k + k - 1 & \text{otherwise.} \end{cases}$$

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