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Border and tangent cells in bargraphs

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Abstract

A border cell is a cell inside a bargraph that has at least one edge in common with an outside cell. A tangent cell is a cell inside a bargraph which is not a border cell and that has at least one vertex in common with an outside cell. In this paper, we study the generating function for the number of bargraphs according to the area, the number of border cells and the number of tangent cells. In particular, we find the generating function for the number of bargraphs according to the area and the inner site-perimeter (number of border cells) and tangent cells.

Keywords: bargraphs; generating functions; site-perimeter.

2010 Mathematics Subject Classification: 05A18.

1. Introduction

A *composition* of n is a word $\sigma = \sigma_1 \cdots \sigma_m$ over alphabet of positive integers such that $\sigma_1 + \cdots + \sigma_m = n$ (see [6]). The letters of σ are called *parts*. Any composition can be represented as a bargraph which is a column convex polyomino, where the lower edge lies on the x-axis. It is drawn on a regular planar lattice grid and made up of square cells. Thus, the size and the number of parts of σ, namely n and m, are the total number of cells and number columns, called *area* and *width*, in the representing bargraph, respectively. Moreover, the part σ_i is the length of the i-th column in the representing bargraph. For instance, the bargraph of the composition 122343412 of 22 is represented on left side of Figure 1. Let B be any bargraph, the *perimeter* of B is the number of edges on the boundary of B, the *site-perimeter*

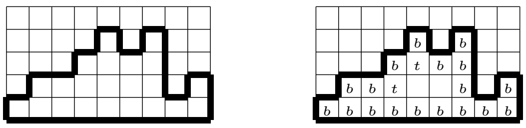

Figure 1: The bargraph 122343412 and border/tangent cells.

of B is the number of nearest-neighbour cells outside the boundary of B, and the *inner site-perimeter* is the number of cells inside B that have at least one edge in common with an outside cell. In [4, 5], the polyominoes according to the area and perimeter were enumerated, while in [3] the site-perimeter of bargraphs was considered, for staircase polygons in [5] and for directed animals in [2]. Recently, Blecher, Brennan and Knopfmacher [1] considered the inner site-perimeter in bargraphs.

We refine the main result of [1] as follows. Let B be any bargraph. A *border cell* of B is a cell inside of B that has at least one edge in common with an outside cell of B. Clearly, the inner site-perimeter of B is the number of border cells of B. A *tangent cell* of B is a cell inside of B which is not a border cell of B and that has at least one vertex in common with an outside cell of B. For instance, in the right side of Figure 1, the border cells are marked by b and the tangent cells are marked by t. Our main result is to enumerate the number of bargraphs according to the area, number of border cells and number of tangent cells.

2. Main results

Let $C(x, y, p, q)$ be the generating function for the number of bargraphs according to the area, width, number of border cells and number of tangent cells marked by x, y, p and q, respectively.

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In order to find a formula for $C(x,y,p,q)$, we need the following notation. An x-bargraph is a bargraph remaining after removing the edges on x-axis. For instance, Figure 2 represents the x-bargraph B of the bargraph on right side of Figure 1, where B has 22 cells, 12 border cells, and 4 tangent cells. We define $\overline{C}(x,y,p,q)$ to be the generating function for the number of x-bargraphs according to the area, width, number of border cells and number of tangent cells.

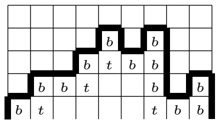

Figure 2: The x-bargraph 122343412 and border/tangent cells.

Define $\mathcal{B}_d$ to be the set of all bargraphs in $\mathcal{B}$ such that the length of each column is at least d. Note that each bargraph B with m parts 1 can be represented as $B = B'1B'\cdots 1B'$ such that $B' \in \mathcal{B}_2$. Thus,

$$C(x,y,p,q) = \sum_{m\geq 0} x^m y^m p^m (\overline{C}(x,pxy,p,q))^{m+1} = \frac{\overline{C}(x,pxy,p,q)}{1 - pxy\overline{C}(x,pxy,p,q)}. \tag{1}$$

In order to write an equation for $\overline{C}(x,y,p,q)$, we define three subsets of x-bargraphs. A *left x-bargraph* is an x-bargraph where its bottom leftmost vertical edge (if exists) is removed (we consider that the border does not include this edge and its vertices). Similarly, we define *right x-bargraph* and *left-right x-bargraph*. Let $D_L(x,y,p,q)$, $D_R(x,y,p,q)$, and $D_{LR}(x,y,p,q)$ be the generating functions for the number of left, right, left-right x-bargraphs according to the area, width, number of border cells and number of tangent cells, respectively.

Note that each x-bargraph B can be decomposed as $B^{(1)}1B^{(2)}\cdots 1B^{(d+1)}$ where $B^{(j)}$ is an x-bargraph such that the length of each column is at least two. Clearly $B^{(1)}$ is a right x-bargraph, $B^{(d+1)}$ is left a x-bargraph, and $B^{(j)}$ is a left-right x-bargraph for all $j = 2,3,\ldots,d$. Thus,

$$\overline{C}(x,y,p,q) = D(x,y,p,q) + \sum_{d\geq 1} (pxy)^d D_L(x,y,p,q) D_R(x,y,p,q) (D_{LR}(x,y,p,q))^{d-1},$$

which leads to

$$\overline{C}(x,y,p,q) = D(x,y,p,q) + \frac{pxy D_L(x,y,p,q) D_R(x,y,p,q)}{1 - pxy D_{LR}(x,y,p,q)}, \tag{2}$$

where $D(x,y,p,q)$ is the generating function for the number x-bargraphs where the length of each column is at least two, according to the area, width, number of border cells and number of tangent cells.

Define α to be a map from x-bargraphs (left x-bargraphs, right x-bargraphs, left-right x-bargraphs) such that the length of each column is at least two to x-bargraphs as follows. For given B, any x-bargraph (left, right or left-right x-bargraph) where the length of each column is at least two, (Step 1) we add all the removed edges that lie or touch the x-axis to get B' (so the obtained result is just a bargraph), (Step 2) we remove the bottom cell in each column of B' to get B'', and at the end (Step 3) we remove the edges of B'' on the x-axis on the resulting bargraph obtained in Step 2. The result is defined as $\alpha(B)$.

By the map α, we obtain

$$D(x,y,p,q) = 1 + \frac{p^2x^2y}{1-px} + p^2\left(\overline{C}(x,xy,p,q) - 1 - \frac{px^2y}{1-px}\right),$$

where 1 counts the empty x-bargraph, $\frac{p^2x^2y}{1-px}$ counts all the x-bargraphs with one column of length at least two, and the last term counts all the x-bargraphs with at least two columns, where the length of each column is at least two.

By using the map α on left, right or left-right x-bargraphs, we obtain

$$D_L(x,y,p,q) = D_R(x,y,p,q) = 1 + \frac{p^2x^2y}{1-px} + pq\left(\overline{C}(x,xy,p,q) - 1 - \frac{px^2y}{1-px}\right)$$

and

$$D_{LR}(x,y,p,q) = 1 + \frac{pqx^2y}{1-px} + q^2\left(\overline{C}(x,xy,p,q) - 1 - \frac{px^2y}{1-px}\right).$$

Define $W(y) = \frac{x^2y}{1-px}$. By substituting the expressions of D, D_L, D_R and D_{LR} into (2), we obtain

$$\overline{C}(x,y,p,q) = 1 + p^2W(y) + p^2\left(\overline{C}(x,xy,p,q) - 1 - pW(y)\right)$$

$$+\frac{pxy\left(1+p^2W(y)+pq\left(\overline{C}(x,xy,p,q)-1-pW(y)\right)\right)^2}{1-pxy\left(1+pqW(y)+q^2\left(\overline{C}(x,xy,p,q)-1-pW(y)\right)\right)}. \tag{3}$$

By (1) we have our main result for this section.

Theorem 2.1. *The generating function $C(x,y,p,q)$ is given by*

$$C(x,y,p,q)=\frac{\overline{C}(x,pxy,p,q)}{1-pxy\overline{C}(x,pxy,p,q)},$$

where $\overline{C}(x,y,p,q)$ satisfies (3).

For instance, (3) with $p=q=1$ gives

$$\overline{C}(x,y,1,1)=\overline{C}(x,xy,1,1)+\frac{xy(\overline{C}(x,xy,1,1))^2}{1-xy\overline{C}(x,xy,1,1)}=\frac{\overline{C}(x,xy,1,1)}{1-xy\overline{C}(x,xy,1,1)},$$

where it is not hard to see that $\overline{C}(x,y,1,1)=\frac{1-x}{1-x-xy}$, as expected. Thus by Theorem 2.1 for $p=q=1$, we have

$$C(x,y,1,1)=\frac{1-x}{1-x-x^2y-xy(1-x)}=\frac{1-x}{1-x-xy},$$

as expected.

Example 2.1. *By taking (3) with $q=1$, we obtain*

$$\overline{C}(x,y,p,1)=1-p^2+p^2(1-p)W(y)+\frac{pxy(1-p)^2+p^2(1+2(1-p)xy)\overline{C}(x,xy,p,1)}{1-pxy\overline{C}(x,xy,p,1)}.$$

Note that $\overline{C}(x,y,p,1)$ can be represented as a continued fraction, which we leave for the interested reader.

Define $f(x,y)=\frac{\partial}{\partial p}\overline{C}(x,y,p,1)\mid_{p=1}$. By differentiating $\overline{C}(x,y,p,1)$ with respect to p and then substituting $p=1$, we obtain

$$f(x,y)=-2-W(y)+\frac{(2-2xy)\overline{C}(x,xy,1,1)+f(x,xy)-xy(3-2xy)\overline{C}(x,xy,1,1)^2}{(1-xy\overline{C}(x,xy,1,1))^2},$$

which, by using $\overline{C}(x,y,1,1)=\frac{1}{1-xy/(1-x)}$, implies

$$f(x,y)=\frac{xy((1-x)^2-x^3y^2)}{(1-x-xy)^2(1-x)}+\frac{(1-x-x^2y)^2}{(1-x-xy)^2}f(x,xy). \tag{4}$$

Thus, by iterating it infinitely many times (here we assume that $|x|<1$), we get

$$\frac{\partial}{\partial p}\overline{C}(x,y,p,1)\mid_{p=1}=\frac{xy(1-x-x^3+x^4-x^3y^2)}{(1-x-xy)^2(1-x)^2(1+x+x^2)}.$$

Theorem 2.1 with using (4) and $\overline{C}(x,y,1,1)=1/(1-xy/(1-x))$ gives

$$\frac{\partial}{\partial p}C(x,y,p,1)\mid_{p=1}=\frac{xy(1-x-x^3+x^4-x^6y^2)}{(1-x-xy)^2(1-x)(1-x^3)}.$$

In particular,

$$\frac{\partial}{\partial p}C(x,1,p,1)\mid_{p=1}=\frac{x(1-x-x^3+x^4-x^6)}{(1-2x)^2(1-x)(1-x^3)},$$

which gives that the total number of border cells (inner site-perimeter) over all bargraphs with n cells is given by $\frac{27}{28}n+\frac{13}{49}+O(n/2^n)$, as shown in [1].

As a consequence of Theorem 2.1, we can find the total of the number tangent cells in bargraphs with n cells. More precisely, (3) with $p=1$ gives

$$\overline{C}(x,y,1,q)=\overline{C}(x,xy,1,q)+\frac{xy\left(1-q+(1-q)W(y)+q\overline{C}(x,xy,1,q)\right)^2}{1-xy\left(1-q^2+q(1-q)W(y)+q^2\overline{C}(x,xy,1,q)\right)}.$$

Define $g(x,y)=\frac{\partial}{\partial q}\overline{C}(x,y,1,q)\mid_{q=1}$. By differentiating $\overline{C}(x,y,1,q)$ with respect to q and then substituting $q=1$, we obtain

$$g(x,y)=\frac{(1+x)x^4y^3}{(1-x-xy)^2}+\frac{(1-x-x^2y)^2}{(1-x-xy)^2}g(x,xy).$$

Thus, by iterating it infinitely many times (here we assume that $|x| < 1$), we get

$$g(x,y) = \frac{\partial}{\partial q}\overline{C}(x,y,1,q)\mid_{q=1} = \frac{(1+x)x^4y^3}{(1-x-xy)^2(1-x^3)}.$$

By Theorem 2.1 and expression of $g(x,y)$, we have

$$\frac{\partial}{\partial q}C(x,y,1,q)\mid_{q=1} = \frac{(1+x)x^7y^3}{(1-x-xy)^2(1-x^3)}.$$

Corollary 2.1. *The average number of the tangent cells over all bargraphs with n cells is given by $\frac{3n}{112} - \frac{71}{392} + O(1/2^n)$.*

Thus, by Example 2.1 and the above corollary, we obtain that the average number of the border and tangent cells over all bargraphs with n cells is given by $\frac{111n}{112} + \frac{33}{392} + O(1/2^n)$.

We end this paper by noting two applications of Theorem 2.1 as follows. By applying Theorem 2.1 with $x = 1$ and $p = q = t$, we obtain that the generating function $\overline{C}(1,y,t,t)$ for the number of x-bargraphs according to the width and the number of border and tangent cells is given by the following equation

$$\overline{C}(1,y,t,t) = \frac{1-t^2(1-y)+t^2\overline{C}(1,y,t,t)}{1-ty+t^3y(1-y)-yt^3\overline{C}(1,y,t,t)},$$

which leads to

$$1-t^2(1-y)-(1-ty-t^2+t^3y(1-y))\overline{C}(1,y,t,t)+yt^3(\overline{C}(1,y,t,t))^2 = 0.$$

Hence, the generating function $\overline{C}(1,y,t,t)$ is given by

$$\frac{1-t^2-ty(1-t^2(1-y))-\sqrt{(1-t^2-ty(1-t^2(1-y)))^2-4yt^3(1-t^2(1-y))}}{2yt^3}.$$

In particular (see Sequence A004148 in [7]), we have

$$\overline{C}(1,y,t,t) = \frac{1-t-t^2-\sqrt{(1-t-t^2)^2-4t^3}}{t^3} = \frac{1}{1-t-t^2}F\left(t^3/(1-t-t^2)^2\right),$$

where $F(t) = \frac{1-\sqrt{1-4t}}{2t}$ is the generating function for the Catalan numbers. So the generating function for the number of x-bargraphs according to the width and the number of border and tangent cells is given by $\overline{C}(1,y,t,t)/(1-ty\overline{C}(1,y,t,t))$.

By applying Theorem 2.1 with $y = q = x = 1$ and $p = t$, we obtain that the generating function $\overline{C}(1,1,t,1)$ for the number of x-bargraphs according to the number of border cells (inner site-perimeter) is given by

$$\frac{1+t-3t^2+2t^3-\sqrt{1-2t-9t^2+6t^3+9t^4-12t^5+4t^6}}{2t}.$$

Thus, the generating function $C(1,1,t,1)$ for the number of bargraphs according to the number of border cells (inner site-perimeter) is given by $\overline{C}(1,1,t,1)/(1-t\overline{C}(1,1,t,1))$.

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