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On the generalization of harmonic graphs

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Abstract

A possible generalization of irregular harmonic graphs is presented. Starting with the study of known properties of traditional harmonic graphs, we introduce the notion of almost harmonic graphs including irregular harmonic graphs as a subset. It is demonstrated on examples that the spectral radius $\rho(G)$ of an almost harmonic graph G can be estimated by simple formulas involving Zagreb indices. It is verified that if G is an almost regular graph with m edges, then $\rho(G) \geq (M_2/m)^{1/2}$, where M_2 is the second Zagreb index of G.

Keywords: harmonic graph; irregular harmonic graph; almost harmonic graph; spectral radius; Zagreb indices.

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1. Introduction

In this study, we consider connected simple graphs. For a graph G, $V(G)$ and $E(G)$ denote the sets of vertices and edges, respectively, in G. Denote by n and m the numbers of vertices and edges, respectively, in G. For a vertex u in G, $d(u)$ stands for the degree of u. By the *degree set* of a graph G, we mean the set of all different vertex degrees of G. We denote by $\Delta = \Delta(G)$ and $\delta = \delta(G)$ the maximum and minimum degrees, respectively, of G. An edge of G connecting the vertices u and v is denoted by uv. By an m-edge graph, we mean a graph with m edges. For a graph G, let $\{m_{r,s} = m_{r,s}(G) : m_{r,s} > 0,\ 1 \leq s, r \leq \Delta\}$ be the finite set of positive integers $m_{r,s}$ representing the number of edges in G with end-vertex degrees r and s. For simplicity, numbers $m_{r,s}$ are called the edge-parameters of G.

We use the standard terminology of graph theory, for notations not defined here we refer the reader to [3,6]. Denote by $A(G)$ the adjacency matrix of graph G. The largest eigenvalue $\rho(G)$ of $A(G)$ is called the spectral radius of G.

A graph is called R-regular if all its vertices have the same degree R. A connected graph is called irregular if it contains at least two vertices with different degrees. Bidegreed (respectively, tridegreed) graph is an irregular graph whose degree set contains exactly two (respectively, three) elements. A connected bipartite bidegreed graph G is *semiregular* if every edge of G joins a vertex of degree δ to a vertex of degree Δ. A connected graph G is called *harmonic* (pseudo-regular) [4, 13, 15, 17] if there exists a positive integer $p(G)$ such that each vertex u of G has the same average neighbor degree number equal to $p(G)$. For the spectral radius of a harmonic graph G, the equality $\rho(G) = p(G)$ holds. From the definition, it follows that any regular graph is a harmonic graph. Irregular harmonic graphs are called *strictly harmonic* graphs. It is easy to see that there exist infinitely many bipartite and non-bipartite harmonic graphs [4, 9, 13, 15, 17]. A bipartite graph G is called *pseudo-semiregular* [17] if each vertex in the same part of bipartition has the same average degree. From these definitions, it follows that any semiregular graph is a bipartite pseudo-semiregular graph.

In this note, a possible generalization of irregular harmonic graphs is presented. Starting with the study of known properties of traditional harmonic graphs, we introduce the notion of almost harmonic graphs containing, as a subset, the irregular harmonic graphs. Methods are outlined for the construction of infinite sequence of almost harmonic graphs. Additionally, it is demonstrated on examples that the spectral radius $\rho(G)$ of an almost harmonic graph G can be estimated by simple formulas containing Zagreb indices.

2. Some properties of strictly harmonic graphs

Let $M_1 = M_1(G)$ and $M_2 = M_2(G)$ be the first and second Zagreb indices of a graph G. By definition, the first Zagreb index M_1 is equal to the sum of squares of the degrees of the vertices of G, and the second Zagreb index M_2 is equal

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to the sum of products of the degrees of pairs of adjacent vertices of the graph G [1, 7, 8, 10–12, 16, 18], that is

$$M_1 = M_1(G) = \sum_{u \in V(G)} \big(d(u)\big)^2 \quad \text{and} \quad M_2 = M_2(G) = \sum_{uv \in E(G)} d(u)d(v).$$

Lemma 2.1. [18] *Let G be a connected graph and let $\rho(G)$ be the spectral radius of G. Then*

$$\rho(G) \geq \frac{2M_2}{M_1},$$

and equality holds if and only if G is regular or strictly harmonic.

Lemma 2.2. [5, 14, 18] *Let G be a strictly harmonic m-edge graph with spectral radius $\rho(G)$. Then*

$$\rho(G) = \frac{2M_2}{M_1} = \sqrt{\frac{M_2}{m}} = \frac{M_1}{2m}.$$

3. Almost harmonic graphs

Let G be a connected graph and consider the graph invariant defined as

$$\Omega(G) = \big(M_1\big)^2 - 4mM_2\,.$$

It is observed that $\Omega(G)$ may be positive, negative or zero. From this, it follows that connected graphs can be classified into 3 disjoint subsets. Examples of such graphs are demonstrated in Figure 1. For graphs depicted in Figure 1, it holds that $\Omega(J_1) = 0$, $\Omega(J_2) = 0$, $\Omega(J_3) = 28$ and $\Omega(J_4) = -24$.

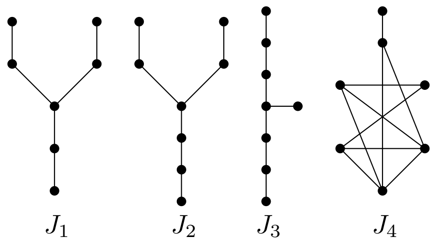

Figure 1: Connected graphs characterized by their topological indices $\Omega(G)$.

A connected graph G is said to be an *almost harmonic graph* if $\Omega(G) = 0$ holds. It is easy to see that J_1 is a 7-vertex strictly harmonic graph, while J_2 is an 8-vertex almost harmonic graph.

Proposition 3.1. *Let G be a connected m-edge graph characterized by the corresponding first and second Zagreb indices. Then G is an almost harmonic graph if*

$$\frac{2M_2}{M_1} = \sqrt{\frac{M_2}{m}} = \frac{M_1}{2m}\,.$$

Proof. From the given condition, it follows directly that $\Omega(G) = 0$. □

From Proposition 3.1 and Lemmas 2.1 and 2.2, the next result follows.

Corollary 3.1. *For an almost harmonic graph G, it holds that*

$$\rho(G) \geq \frac{2M_2}{M_1} = \sqrt{\frac{M_2}{m}} = \frac{M_1}{2m}$$

with equality if G is strictly harmonic graph.

It can be concluded that strictly harmonic graphs represent a subset of the set of almost harmonic graphs.

4. Estimation of spectral radius

In what follows, it will be demonstrated that the topological index

$$\Lambda(G) = \sqrt{\frac{M_2}{m}}$$

can be considered as a very good estimate (lower bound) of the spectral radius of almost harmonic graphs. According to our observations based on computational results, for almost harmonic graphs the difference $\varepsilon = \rho(G) - \Lambda(G)$ is considerably small.

Lemma 4.1. *Let $G(\Delta, \delta)$ be a connected bidegreed m-edge graph, for which $m_{\delta,\delta} = m_{\Delta,\Delta} = m/4$ is fulfilled. Then, $G(\Delta, \delta)$ is an almost harmonic graph.*

Proof. Because $m_{\delta,\Delta} = m - (m_{\delta,\delta} + m_{\Delta,\Delta}) = \frac{m}{2}$, this implies that

$$M_1 = 2\delta m_{\delta,\delta} + (\delta + \Delta)m_{\delta,\Delta} + 2\Delta m_{\Delta,\Delta} = (\delta + \Delta)m$$

and

$$M_2 = \delta^2 m_{\delta,\delta} + \Delta\delta m_{\delta,\Delta} + \Delta^2 m_{\Delta,\Delta} = \frac{m}{4}(\Delta + \delta)^2.$$

After performing simple computations, we obtain $\Omega(G(\Delta, \delta)) = M_1^2 - 4mM_2 = 0$.

□

Example 4.1. *Consider the non-isomorphic, bidegreed graphs shown in Figure 2. It is easy to see that polyhedral graphs G_a, G_b and G_c are characterized by several identical topological parameters: $n = 7$, $m = 12$, $M_1 = 84$ and $M_2 = 147$. Because for edge-parameters the equality $m_{3,3} = m_{4,4} = m/4 = 3$ holds, from Lemma 4.1 it follows that these graphs are almost harmonic and $\Lambda(G_a) = \Lambda(G_b) = \Lambda(G_c) = 7/2$ is fulfilled for them. The corresponding spectral radii are: $\rho(G_a) = 3.514137$, $\rho(G_b) = 3.507903$ and $\rho(G_c) = 3.503224$. As we can conclude, the values of corresponding spectral radii are very close to the parameter $\Lambda = 3.5$.*

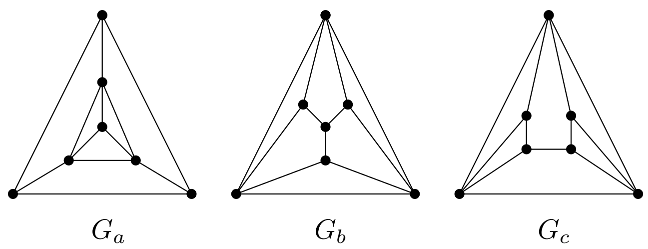

Figure 2: Almost harmonic polyhedral graphs.

Remark 4.1. *It should be noted that $\Lambda(G)$ is a good estimate (approximation) of spectral radius not only for almost harmonic graphs, but also for several connected graphs differing from almost harmonic graphs. Recently, It has been proved [2, 5] that there exists a broad class of connected graphs (the so-called Z_2 graphs) including harmonic, semiregular, pseudo-semiregular graphs, for which the equality $\rho(G) = \Lambda(G)$ holds.*

Lemma 4.2. *Let $G(\Delta, \delta)$ be a connected bidegreed graph with m edges, such that the equality*

$$\frac{m_{\delta,\Delta}}{m} = \frac{m_{\Delta,\Delta}}{m} = \frac{4}{9}$$

holds. Then, $G(\Delta, \delta)$ is an almost harmonic graph.

Proof. It suffices to verify that

$$M_1^2 - 4mM_2 = (\Delta - \delta)^2(9m_{\Delta,\Delta} - 4m)m_{\Delta,\Delta}.$$

Because $m_{\delta,\Delta} = m_{\Delta,\Delta}$, this implies that $m_{\delta,\delta} = m - 2m_{\Delta,\Delta}$. Consequently,

$$M_1 = 2\delta m_{\delta,\delta} + (\delta + \Delta)m_{\delta,\Delta} + 2\Delta m_{\Delta,\Delta} = 2m\delta + 3(\Delta - \delta)m_{\Delta,\Delta}$$

and

$$M_2 = \delta^2 m_{\delta,\delta} + \Delta\delta m_{\delta,\Delta} + \Delta^2 m_{\Delta,\Delta} = m\delta^2 + (\Delta^2 + \Delta\delta - 2\delta^2)m_{\Delta,\Delta}.$$

Moreover, because

$$M_1^2 = 4\delta^2 m^2 + 12\delta m(\Delta - \delta)m_{\Delta,\Delta} + 9(\Delta - \delta)^2 m_{\Delta,\Delta}^2$$

and

$$4mM_2 = 4m^2\delta^2 + 4m(\Delta^2 + \Delta\delta - 2\delta^2)m_{\Delta,\Delta},$$

we have

$$\Omega(G) = M_1^2 - 4mM_2 = (\Delta - \delta)^2(9m_{\Delta\Delta} - 4m)m_{\Delta,\Delta}.$$

□

Analogously to Lemma 4.2, the following result can be obtained.

Lemma 4.3. *Let $G(\Delta, \delta)$ be a connected bidegreed graph with m edges, for which the equality*

$$\frac{m_{\delta,\Delta}}{m} = \frac{m_{\delta,\delta}}{m} = \frac{4}{9}$$

holds. Then, $G(\Delta, \delta)$ is an almost harmonic graph.

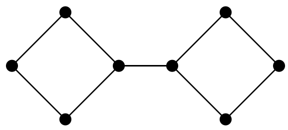

Figure 3: Almost harmonic bicyclic graph B_8.

Example 4.2. *Consider the 8-vertex and 9-edge graph B_8 depicted in Figure 3. The spectral radius of the graph B_8 is $\rho(B_8) = 2.3429$. Because for the 9-edge graph B_8, the equality $m_{2,3}/m = m_{2,2}/m = 4/9$ holds, from Lemma 4.3 it follows that B_8 is an almost harmonic graph with $\Lambda(B_8) = 7/3 = 2.3333$. It can be seen that the difference $\varepsilon = \rho(B_8) - \Lambda(B_8)$ is considerably small.*

Proposition 4.1. *There exist infinitely many almost harmonic trees satisfying $\Lambda = 2$.*

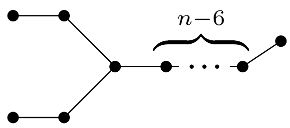

Figure 4: An infinite sequence of almost harmonic trees T_n.

Proof. In Figure 4, a tree T_n with $n \geq 8$ vertices is depicted. For this infinite sequence of trees T_n with degree set $\{1, 2, 3\}$ one obtains that $m_{1,2} = 3$, $m_{2,2} = n - 7$, $m_{2,3} = 3$, $m = n - 1$ and $M_1 = M_2 = 4(n - 1)$. It is easy to see that if $n \geq 8$ then T_n are almost harmonic graphs characterized by the following relationship:

$$\Lambda(T_n) = \frac{M_1}{2m} = \sqrt{\frac{M_2}{m}} = \frac{2M_2}{M_1} = 2 < \rho(T_n).$$

□

Example 4.3. *For the 8-vertex tree T_8 (see Figure 4), it holds that $m_{1,2} = 3$, $m_{2,2} = 1$, $m_{2,3} = 3$ and $M_1 = M_2 = 28$. Consequently, we have $\rho(T_8) = 2.02852 > 2 = \Lambda(T_8)$.*

Proposition 4.2. *There exist infinitely many almost harmonic cyclic graphs with identical Λ parameter.*

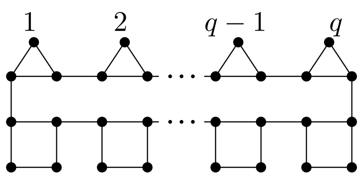

Figure 5: Bidegreed almost harmonic graph $G(q)$ where q is an arbitrary positive integer.

Proof. Consider the infinite sequence of connected bidegreed graphs $G(q)$ depicted in Figure 5 with degree set $\{2,3\}$, where q is an arbitrary positive integer. These planar graphs $G(q)$ contain q triangles, q quadrilaterals, and one $4q$-gon. It is easy to see that for these graphs $n = 7q$, $m = 9q$, $m_{2,2} = q$, $m_{2,3} = 4q$, $m_{3,3} = 4q$, $M_1 = 48q$ and $M_2 = 64q$ are fulfilled. Because $m_{2,3}/m = m_{3,3}/m = 4/9$, from Lemma 4.2, it follows that graphs $G(q)$ are almost harmonic with identical parameter $\Lambda(G(q)) = 8/3$. □

Proposition 4.3. *There exist infinitely many almost harmonic tridegreed graphs having identical Λ parameter, where Λ is a positive integer.*

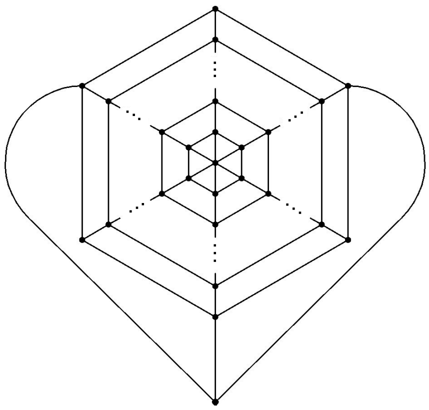

Figure 6: Planar almost harmonic graph $H(k)$ where $k \geq 2$ is an arbitrary positive integer.

Proof. Consider the infinite sequence of connected tridegreed graphs $H(k)$ shown in Figure 6, with degree set $\{3,4,6\}$, where $k \geq 2$ is an arbitrary positive integer. These planar graphs contain only triangles and quadrilaterals. It is easy to check that the graphs $H(k)$ are not strictly harmonic, but they are almost harmonic. For these graphs $n = 6k+2$, $m = 2n-1$, $m_{3,4} = 12$, $m_{4,4} = 2n-19$, $m_{4,6} = 6$, $M_1 = 16n-8$ and $M_2 = 32n-16$ are fulfilled. This implies that $\Omega(H(k)) = M_1^2 - 4mM_2 = 0$, and consequently $\Lambda(H(k)) = 4$. □

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